“Sinyaller ve Sistemler Matematik”

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Aritmetik İşlem nedir?

- Aritmetik; sayılar arasındaki ilişkiler ile sayıların problem çözmede kullanımı ile ilgilenen matematiğin dalıdır. Aritmetik kavramı ile genellikle sayılar teorisi, ölçme ve hesaplama (toplama, çıkarma, çarpma, bölme, üs alma, kök alma, logaritma, trigonometri) kastedilir.
- Aritmetik sistemler, işlem yapılan sayıların türüne göre çeşitlenir. Tam sayı aritmetiği sadece pozitif ve negatif doğal sayılar ile yapılan hesaplamaları içerirken, rasyonel sayı aritmetiği, tam sayılar arasındaki kesirlerle yapılan işlemleri kapsar. Reel sayı aritmetiği, rasyonel ve irrasyonel sayıların her ikisiyle yapılan hesaplamaları barındırır ve tam bir sayı doğrusunu kapsar.
- Sistemlerin ayrımı, kullanılan sayı sistemine göre de yapılır. Ondalık aritmetik, en yaygın kullanılan sistemdir ve sayıları ifade etmek için 0'dan 9'a kadar olan temel rakamları ve bu rakamların kombinasyonlarını kullanır. Öte yandan, çoğunlukla bilgisayarlar tarafından kullanılan ikili aritmetik, sayıları 0 ve 1 rakamlarının kombinasyonları olarak temsil eder. Bazı aritmetik sistemler ise sayılardan farklı sembol gösterimleri üzerinde işlem yapar.

Aritmetik işlemlerde kullanılan temel birimler

Temel birimler kullanılmalıdır. Aritmetik işlemlerde birimler aynı olmak zorundadır,

- Saniye
- Gram, Kgram
- Metre
- Hertz (Hz=1/Saniye)
- Watt
- Derece, sıcaklık
- Derece, açı

Aritmetik işlemler yapılırken temel birimler göz önüne alınır.

Örneğin, f=10GHz, d=10km ise f*d=?
10*10=100 dediniz mi! yandınız.
Yapılması gereken, f=10Ghz=10*10^9Hz, d=10km=10*10^3m; 10^10 * 10^4=10^14 Hz-m

Örnek: Vo=5V, Vi=3mV. Aritmetik işlemlerde Birimler aynı olmak zorundadır, Vo=5*10^3 mV, Vi=3mV

Soru

- **Aşağıdaki 4 ile 9 arasındaki soruları,**
- x(t)=75Sin(2π900000000t+60) analog sinyalinde
- **x(t)=ASin(wt+ø)**
- **ørad=ødeg * π/180**
- **w=2πf ise**
- **T=1/f=2π/w**
- **denklemleri kullanarak çözünüz.**
-
- genlik kaç birimdir? 75
- frekans kaç GHz? 1 000 000 000Hz=1GHz
- faz kaç derecedir? 60
- faz kaç radyan? **π/3**
- peryod, T kaç saniyedir? 1/1 000 000 000=10^-9saniye

17 Equations That Changed the World
by Ian Stewart

1.	**Pythagoras's Theorem**	$a^2 + b^2 = c^2$	Pythagoras,530 BC
2.	**Logarithms**	$\log xy = \log x + \log y$	John Napier, 1610
3.	**Calculus**	$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$	Newton, 1668
4.	**Law of Gravity**	$F = G\frac{m_1 m_2}{r^2}$	Newton, 1687
5.	**The Square Root of Minus One**	$i^2 = -1$	Euler, 1750
6.	**Euler's Formula for Polyhedra**	$V - E + F = 2$	Euler, 1751
7.	**Normal Distribution**	$\Phi(x) = \frac{1}{\sqrt{2\pi\rho}} e^{\frac{(x-\mu)^2}{2\rho^2}}$	C.F. Gauss, 1810

8.	**Wave Equation**	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	J. d'Almbert, 1746
9.	**Fourier Transform**	$f(\omega) = \int_{\infty}^{\infty} f(x) e^{-2\pi i x \omega} \mathrm{d}\, x$	J. Fourier, 1822
10.	**Navier-Stokes Equation**	$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$	C. Navier, G. Stokes, 1845
11.	**Maxwell's Equations**	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ $\quad \nabla \cdot \mathbf{H} = 0$ $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}$ $\quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial E}{\partial t}$	J.C. Maxwell, 1865
12.	**Second Law of Thermodynamics**	$\mathrm{d}S \geq 0$	L. Boltzmann, 1874
13.	**Relativity**	$E = mc^2$	Einstein, 1905
14.	**Schrodinger's Equation**	$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$	E. Schrodinger, 1927
15.	**Information Theory**	$H = -\sum p(x) \log p(x)$	C. Shannon, 1949
16.	**Chaos Theory**	$x_{t+1} = k x_t (1 - x_t)$	Robert May, 1975
17.	**Black-Scholes Equation**	$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$	F. Black, M. Scholes, 1990

Trigonometric Identities

Trigonometric Identities

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\sin\theta\sin\alpha = \frac{1}{2}(\cos(\theta - \alpha) - \cos(\theta + \alpha))$$

$$\cos\theta\cos\alpha = \frac{1}{2}(\cos(\theta + \alpha) + \cos(\theta - \alpha))$$

$$\sin\theta\cos\alpha = \frac{1}{2}(\sin(\theta + \alpha) + \sin(\theta - \alpha))$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Trigonometric Formula

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A - \tan^2 A = 1 \qquad \operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2 \sin A \cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

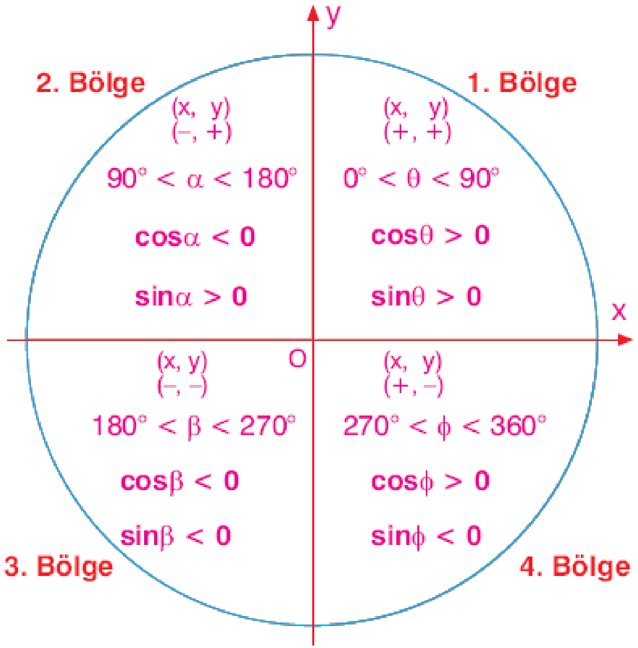
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^3 A = \frac{3 \cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$



	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Tanımsız

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$$

$$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$

$$\sin(\pi-\theta)=\sin\theta$$

$$\sin(\pi+\theta)=-\sin\theta$$

$$\cos(\pi-\theta)=-\cos\theta$$

$$\cos(\pi+\theta)=-\cos\theta$$

Sine and Cosine Addition and Subtraction Formulas

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

$$\sin(a-b)=\sin a\cos b-\cos a\sin b$$

$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

$$\cos(a-b)=\cos a\cos b+\sin a\sin b$$

$$\cos x=\frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

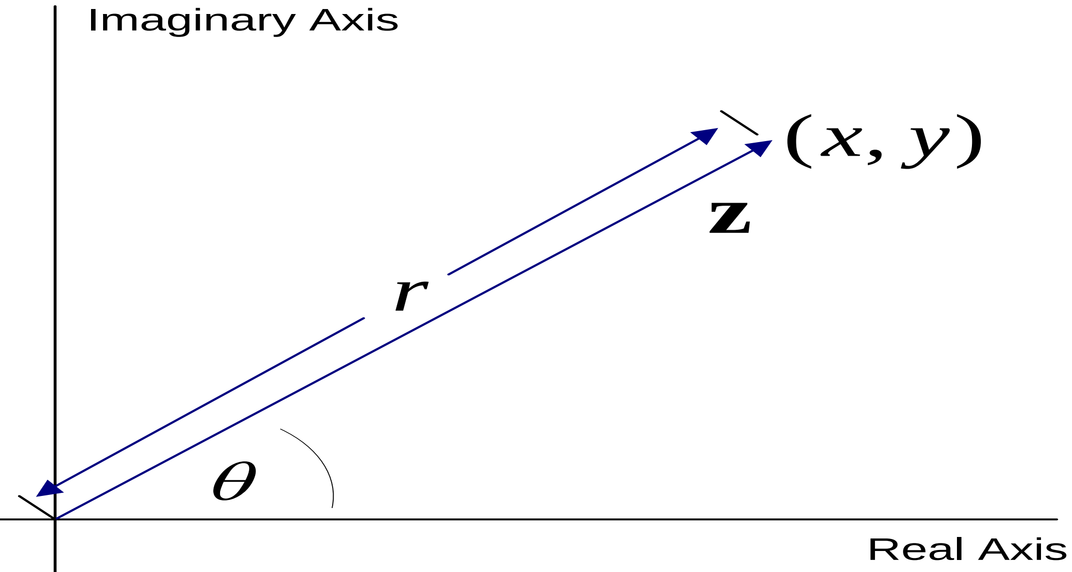
$$\sin x=\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

Complexs Numbers

Form of Complex Number

$$\mathbf{z} = x + iy$$

$$i = \sqrt{-1}$$



Rectangular to Polar:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \operatorname{ang}(\mathbf{z}) = \tan^{-1}\frac{y}{x}$$

Kompleks Düzlemde Analog Sinyal

- Kompleks siyallerde faz bileşeni çok önemlidir.
- $Z=r\exp(j\theta)=r\cos(\theta)+jr\text{Sin}(\theta)$; faz açısı
- $Z(t)=r\exp(jwt)= r\cos(wt)+jr\text{Sin}(wt)$; frekans ve zaman
- $Z(t)=r\exp(jwt+j\theta)= r\cos(wt+\theta)+jr\text{Sin}(wt+\theta)$
- Kararlılık analizinde reel kısım göz önüne alınır, $r\cos(wt+\theta)$

Soru

- Z=rexp(jθ)=r*$e^{j\theta}$ =rcosθ+Jrsinθ
- Burada r: Genlik, θ ise radyan cinsinden faz açısını verir.
- **ødeg=ørad *180/ π**
- J*J=-1
- Z=j10exp(jπ/2) ifadesinde

	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Tanımsız	0	Tanımsız	0
cot	Tanımsız	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	Tanımsız	0	Tanımsız

Sorunun Çözümü

- Genlik değeri nedir?
- 10
- Radyan olarak faz açısının değeri nedir?
- $\pi/2$
- Derece olarak faz açısının değeri nedir?
- 90deg.
- Z ifadesini, Z=a+jb cinsinden yazınız.
- Z=j10*(Cos(90)+jsin(90))
- Bir önceki bulduğunuz değeri kullanarak Z ifadesini sadeleştiriniz.
- Z=j10*j=-10

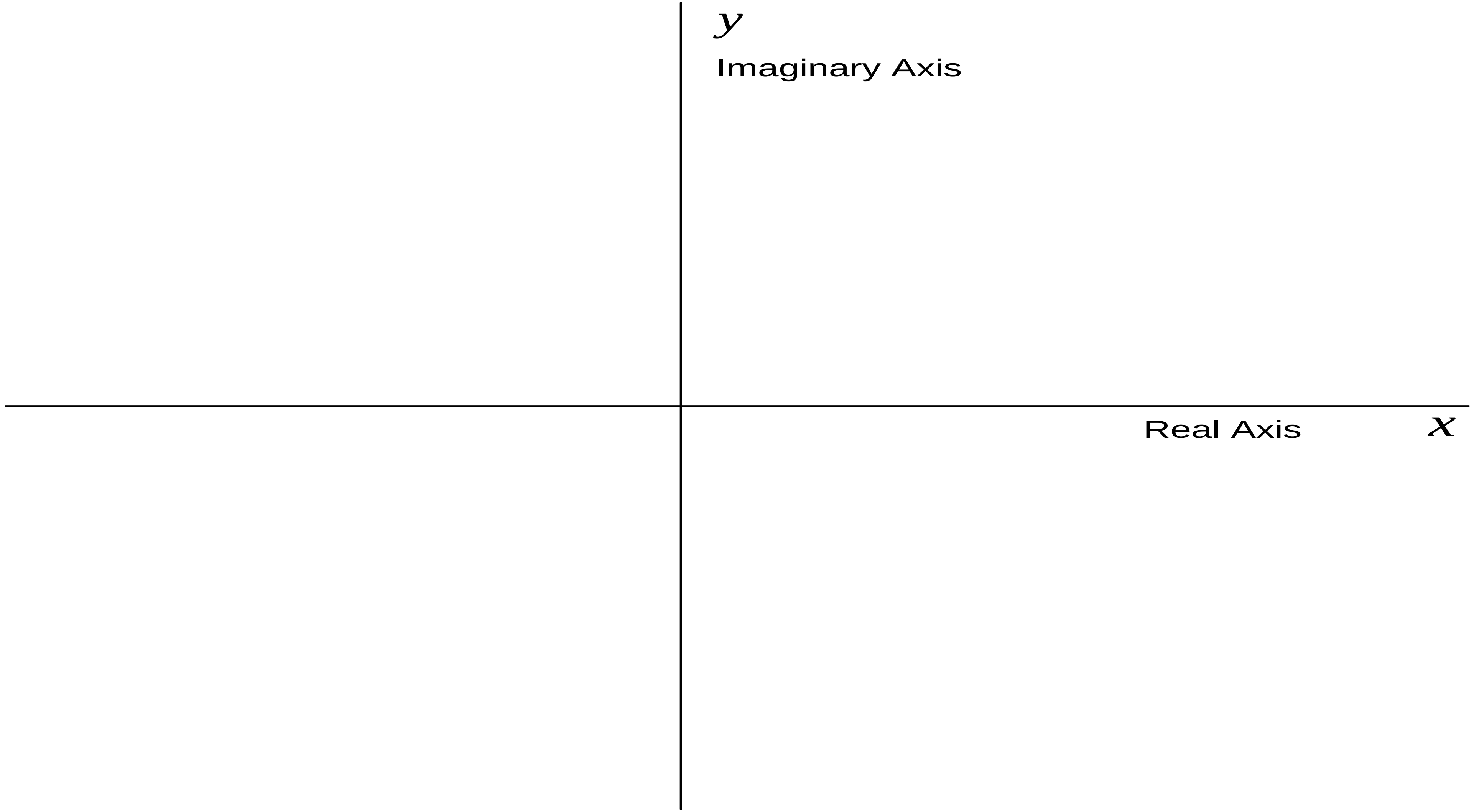
Complex Numbers

- A few operations with complex numbers were used earlier in the text and it was assumed that the reader had some basic knowledge of the subject. The subject will now be approached from a broader perspective and many of the operations will be developed in detail. To deal with Fourier analysis, complex number operations are required.
- In a sense, complex numbers are two-dimensional vectors. In fact, some of the basic arithmetic operations such as addition and subtraction are the same as those that could be performed with the spatial vector forms of Chapter 14 in two dimensions. However, once the process of multiplication is reached, the theory of complex number operations diverges significantly from that of spatial vectors.

Rectangular Coordinate System

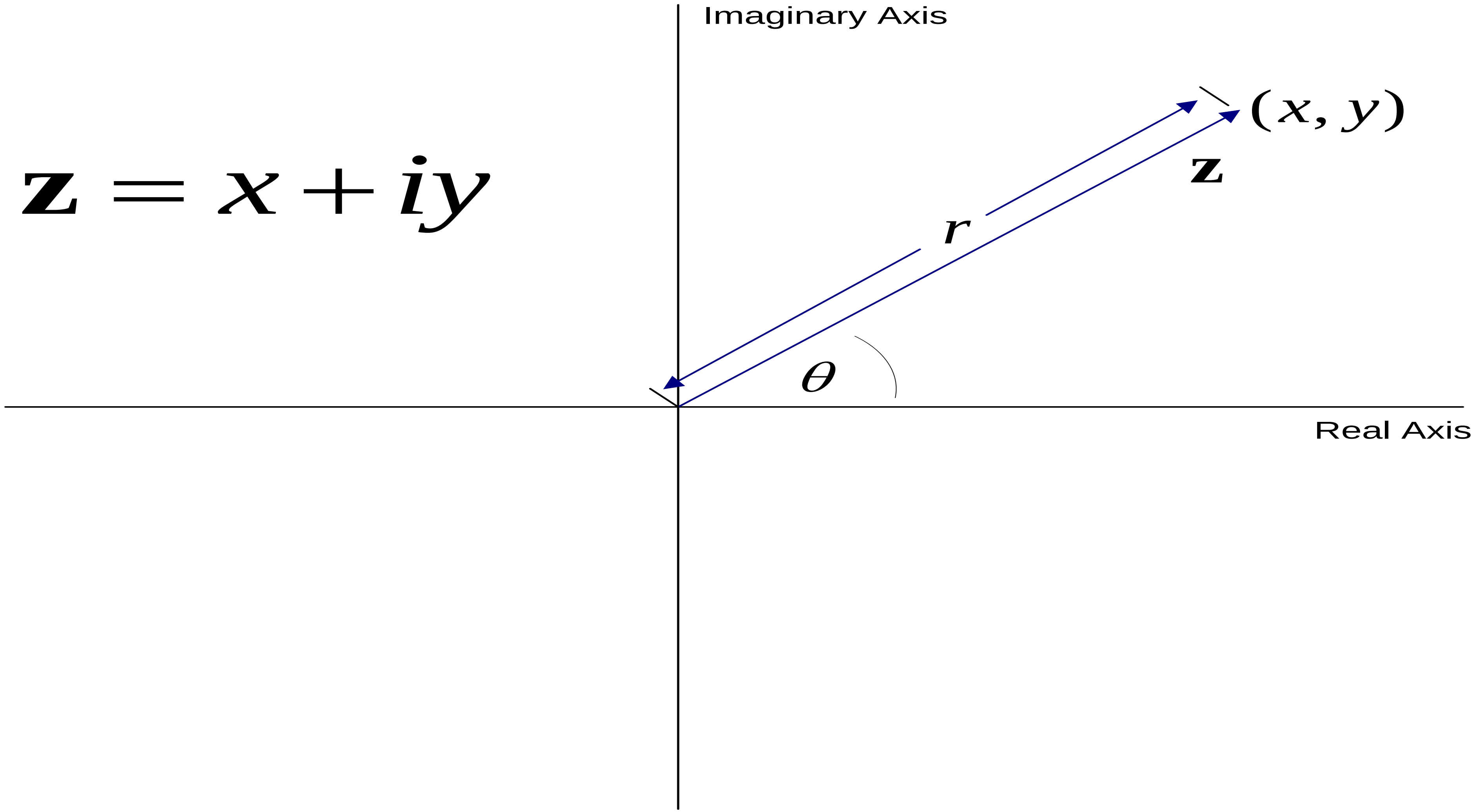
- The development begins with the two-dimensional rectangular coordinate system shown on the next slide. The two axes are labeled as *x* and *y*. In complex variable theory, the ***x*-axis** is called the **real axis** and the ***y*-axis** is called the **imaginary axis**. An imaginary number is based on the definition that follows. $i = \sqrt{-1}$

Complex Plane



Form of Complex Number

$$\mathbf{z} = x + iy$$



Conversion Between Forms

Polar to Rectangular:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Rectangular to Polar:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \mathrm{ang}(\mathbf{z}) = \tan^{-1}\frac{y}{x}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\mathbf{z} = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

$$\mathbf{z} = re^{i\theta}$$

Common Engineering Notation:

$$re^{i\theta} \;\square\; r\angle\theta$$

Example. Convert the following complex number to polar form:

$$\mathbf{z} = 4 + i3$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (3)^2} = 5$$

$$\theta = \tan^{-1}\frac{3}{4} = 36.87^\circ = 0.6435 \text{ rad}$$

$$\mathbf{z} = 5\angle 36.87^\circ \quad \text{or}$$

$$\mathbf{z} = 5e^{i0.6435}$$

Example. Convert the following complex number to polar form:

$$\mathbf{z} = -4 + i3$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + (3)^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{3}{-4}\right) = 180^\circ - \tan^{-1}\frac{3}{4}$$

$$= 180^\circ - 36.87^\circ = 143.13^\circ = 2.498 \text{ rad}$$

$$\mathbf{z} = 5e^{i2.498}$$

Example. Convert the following complex number to rectangular form:

$$\mathbf{z} = 4e^{i2}$$

$$x = 4\cos 2 = -1.6646$$

$$y = 4\sin 2 = 3.6372$$

$$\mathbf{z} = -1.6646 + i3.6372$$

Example. Convert the following complex number to rectangular form:

$$\mathbf{z} = 10e^{-i}$$

$$x = 10\cos(-1) = 5.4030$$

$$y = 10\sin(-1) = -8.4147$$

$$\mathbf{z} = 5.4030 - i8.4147$$

Addition of Two Complex Numbers

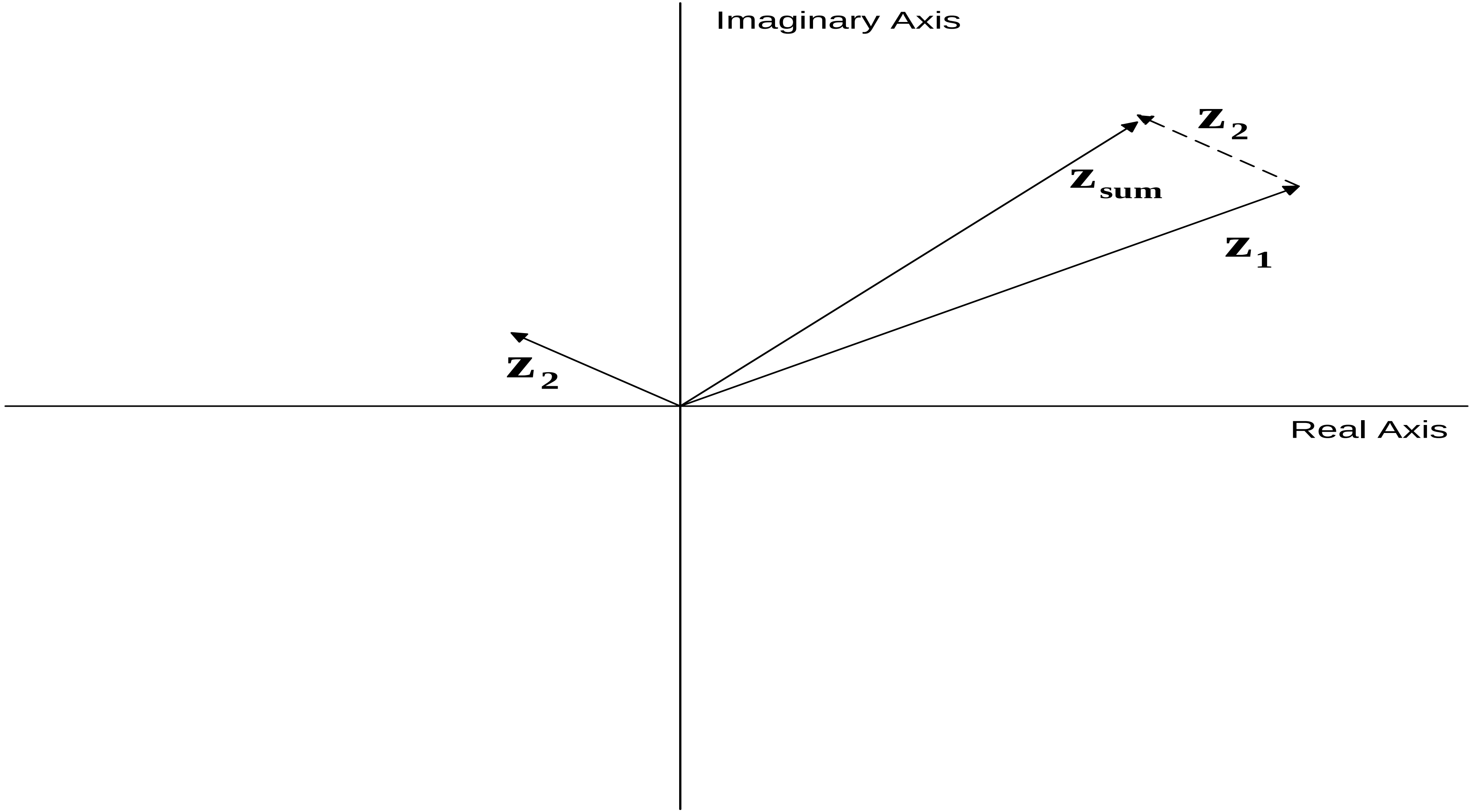
$$\mathbf{z_1} = x_1 + iy_1$$

$$\mathbf{z_2} = x_2 + iy_2$$

$$\begin{aligned}\mathbf{z_{sum}} &= \mathbf{z_1} + \mathbf{z_2} \\ &= x_1 + iy_1 + x_2 + iy_2 \\ &= x_1 + x_2 + i(y_1 + y_2)\end{aligned}$$

A geometric interpretation of addition is shown on the next slide.

Addition of Two Complex Numbers



Subtraction of Two Complex Numbers

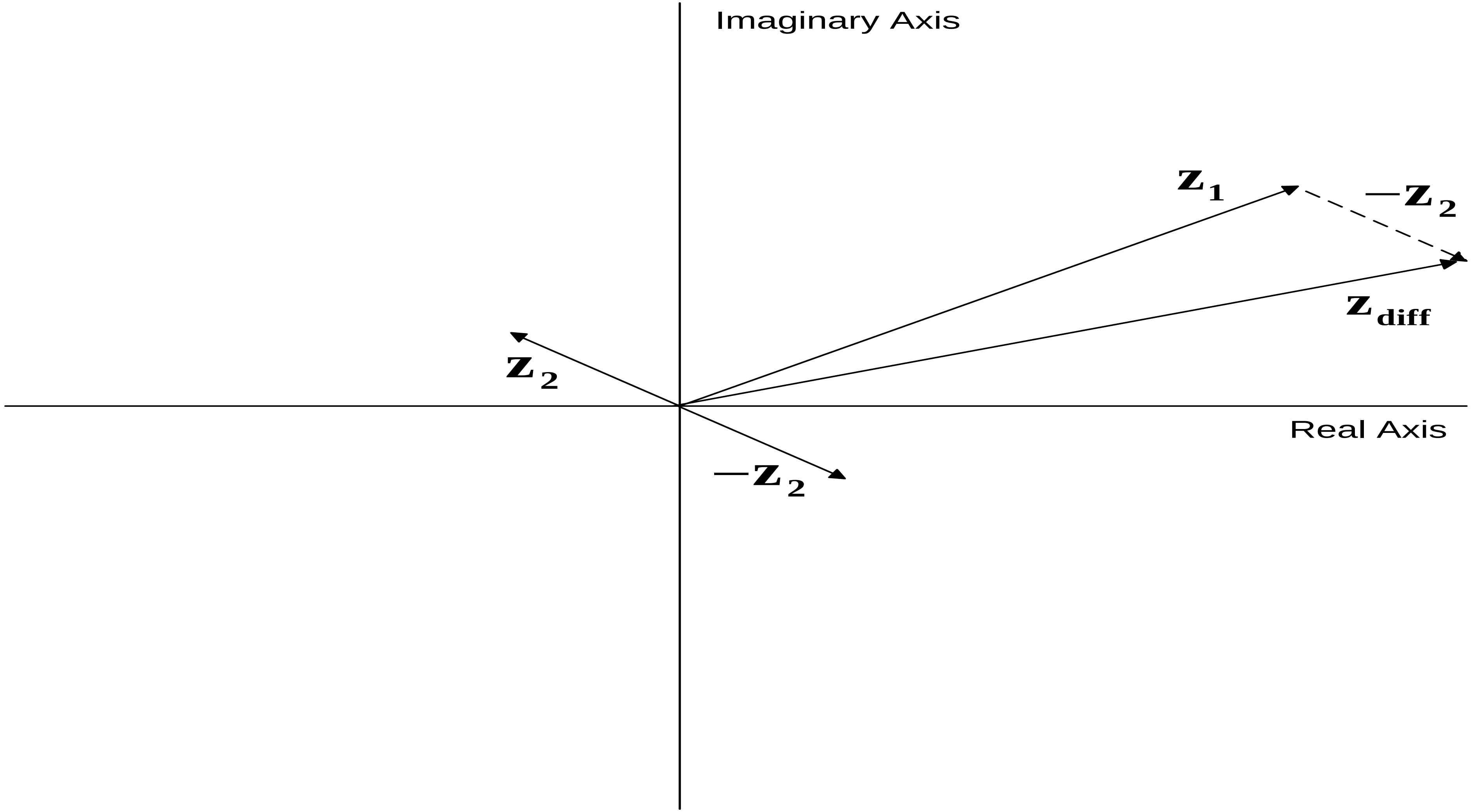
$$\mathbf{z_1} = x_1 + iy_1$$

$$\mathbf{z_2} = x_2 + iy_2$$

$$\begin{aligned}\mathbf{z_{diff}} &= \mathbf{z_1} - \mathbf{z_2} \\ &= x_1 + iy_1 - (x_2 + iy_2) \\ &= x_1 - x_2 + i(y_1 - y_2)\end{aligned}$$

A geometric interpretation of subtraction is shown on the next slide.

Subtraction of Two Complex Numbers



Example. Determine the sum of the following complex numbers:

$$\mathbf{z_1} = 5 + i3$$

$$\mathbf{z_2} = 2 - i7$$

$$\begin{aligned} \mathbf{z_{sum}} &= \mathbf{z_1} + \mathbf{z_2} \\ &= 5 + i3 + 2 - i7 \\ &= 7 - i4 \end{aligned}$$

Example. For the numbers that follow, determine $z_{\text{diff}} = z_1 - z_2$.

$$\mathbf{z_1} = 5 + i3$$

$$\mathbf{z_2} = 2 - i7$$

$$\begin{aligned}\mathbf{z_{diff}} &= \mathbf{z_1} - \mathbf{z_2} \\ &= 5 + i3 - (2 - i7) \\ &= 3 + i10\end{aligned}$$

Multiplication in Polar Form

$$\mathbf{z_1} = r_1 e^{i\theta_1}$$

$$\mathbf{z_2} = r_2 e^{i\theta_2}$$

$$\begin{aligned}\mathbf{z_{prod}} &= \mathbf{z_1 z_2} \\ &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}\end{aligned}$$

Division in Polar Form

$$\mathbf{z_1} = r_1 e^{i\theta_1}$$

$$\mathbf{z_2} = r_2 e^{i\theta_2}$$

$$\mathbf{z_{div}} = \frac{\mathbf{z_1}}{\mathbf{z_2}} = \frac{\left(r_1 e^{i\theta_1}\right)}{\left(r_2 e^{i\theta_2}\right)}$$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Multiplication in Rectangular Form

$$\mathbf{z_1} = x_1 + iy_1$$

$$\mathbf{z_2} = x_2 + iy_2$$

$$\mathbf{z_{prod}} = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$$

$$\mathbf{z_{prod}} = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Complex Conjugate

Start with

$$\mathbf{z} = x + iy = re^{i\theta}$$

The complex conjugate is

$$\overline{\mathbf{z}} = x - iy = re^{-i\theta}$$

The product of z and $\overline{z}$ is

$$(\mathbf{z})(\overline{\mathbf{z}}) = x^2 + y^2 = r^2$$

Division in Rectangular Form

$$\mathbf{z}_{\mathbf{div}} = \frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\mathbf{z}_{\mathbf{div}} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$$= \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{r^2}$$

Example. Determine the product of the following 2 complex numbers:

$$\mathbf{z_1} = 8e^{i2} \qquad \mathbf{z_2} = 5e^{-i0.7}$$

$$\mathbf{z_3} = \mathbf{z_1}\mathbf{z_2} = \left(8e^{i2}\right)\left(5e^{-i0.7}\right) = 40e^{i1.3}$$

$$\begin{aligned}\mathbf{z_3} &= 40(\cos 1.3 + i \sin 1.3) \\ &= 40(0.2675 + i0.9636) \\ &= 10.70 + i38.54\end{aligned}$$

Example. Repeat previous example using rectangular forms.

$$\mathbf{z}_1 = 8e^{i2} = 8(\cos 2 + i\sin 2)$$
$$= 8(-0.4162 + i0.9093) = -3.329 + i7.274$$

$$\mathbf{z}_2 = 5e^{-i0.7} = 5(\cos 0.7 - i\sin 0.7)$$
$$= 5(0.7648 - i0.6442) = 3.824 - i3.221$$

$$\mathbf{z}_3 = \mathbf{z}_1\mathbf{z}_2 = (-3.329 + i7.274)(3.824 - i3.221)$$
$$= -12.73 + 23.43 + i(27.82 + 10.72)$$
$$= 10.70 + i38.54$$

Example. Determine the quotient z_1/z_2 for the following 2 numbers:

$$\mathbf{z_1} = 8e^{i2} \qquad \mathbf{z_2} = 5e^{-i0.7}$$

$$\mathbf{z_4} = \frac{\mathbf{z_1}}{\mathbf{z_2}} = \frac{8e^{i2}}{5e^{-i0.7}} = 1.6e^{i2.7}$$

$$\mathbf{z_4} = 1.6(\cos 2.7 + i \sin 2.7)$$

$$= 1.6(-0.9041 + i0.4274)$$

$$= -1.447 + i0.6838$$

Example. Repeat previous example using rectangular forms.

$$\mathbf{z}_4 = \frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{-3.329 + i7.274}{3.824 - i3.221}$$

$$\mathbf{z}_4 = \frac{(-3.329 + i7.274)}{(3.824 - i3.221)} \frac{(3.824 + i3.221)}{(3.824 + i3.221)}$$

$$= \frac{-12.73 - 23.43 + i(27.82 - 10.72)}{14.62 + 10.37}$$

$$= \frac{-36.16 + i17.10}{25.00} = -1.446 + i0.6840$$

Exponentiation of Complex Numbers: Integer Power

$$\mathbf{z}_{\mathbf{power}} = (\mathbf{z})^N$$

$$\mathbf{z}_{\mathbf{power}} = (re^{i\theta})^N = r^N e^{iN\theta}$$

$$= r^N \cos N\theta + ir^N \sin N\theta$$

$$\cos N\theta = \mathrm{Re}(e^{iN\theta})$$

$$\sin N\theta = \mathrm{Im}(e^{iN\theta})$$

Roots of Complex Numbers

$$\mathbf{z} = re^{i(\theta+2\pi n)}$$

$$\mathbf{z}_{\mathbf{roots}} = \left(re^{i(\theta+2\pi n)}\right)^{1/N} = r^{1/N}e^{i(\theta/N+2\pi n/N)}$$

$$\mathbf{z}_{\mathbf{roots}} = r^{1/N}e^{i\left(\frac{\theta}{N}+\frac{2\pi n}{N}\right)} \text{ for } n = 0,1,2\ldots,N-1$$

$$\mathbf{z}_{\mathbf{principal}} = r^{1/N}e^{i\theta/N}$$

Example. For the value of z below, determine $z_6 = z^6$.

$$\mathbf{z} = 3 + i4 = 5e^{i0.9273}$$

$$\mathbf{z_6} = (\mathbf{z})^6$$

$$= (5e^{i0.9273})^6 = 15,625e^{i5.5638}$$

$$\mathbf{z_6} = 15,625(\cos 5.5638 + i \sin 5.5638)$$

$$= 11,753 - i10,296$$

Example. Determine the 4 roots of $s^4 + 1 = 0$.

$$s^4 = -1 = 1e^{i(\pi + 2\pi n)}$$

$$s = e^{i\left(\frac{\pi}{4} + n\frac{2\pi}{4}\right)} \quad \text{for} \quad n = 0, 1, 2, 3$$

$$s_1 = e^{i\frac{\pi}{4}} = 0.7071 + i0.7071$$

$$s_2 = e^{i\frac{3\pi}{4}} = -0.7071 + i0.7071$$

$$s_3 = e^{i\frac{5\pi}{4}} = -0.7071 - i0.7071$$

$$s_4 = e^{i\frac{7\pi}{4}} = 0.7071 - i0.7071$$

MATLAB Complex Number Operations: Entering in Rectangular Form

- >> z = 3 + 4i
- z =
- 3.0000 + 4.0000i
- >> z = 3 + 4j
- z =
- 3.0000 + 4.0000i
- The i or j may precede the value, but the multiplication symbol (*) is required.

MATLAB Complex Number Operations: Entering in Polar Form

```
>> z = 5*exp(0.9273i)
z = 3.0000 + 4.0000i
>> z = 5*exp((pi/180)*53.13i)
z =3.0000 + 4.0000i
```

This result indicates that polar to rectangular conversion occurs automatically upon entering the number in polar form.

MATLAB Rectangular to Polar Conversion

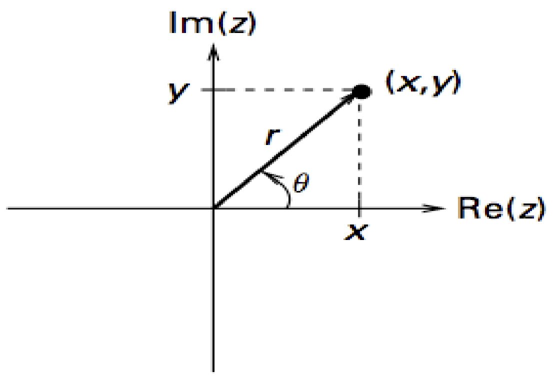
```
>> z = 3 + 4i
z =  3.0000 + 4.0000i
>> r = abs(z)
r = 5
>> theta = angle(z)
theta =   0.9273
```

Other MATLAB Operations

```
>> z_conj = conj(z)
z_conj =   3.0000 - 4.0000i
>>real(z)
ans =     3
>> imag(z)
ans =     4
```

Complex numbers

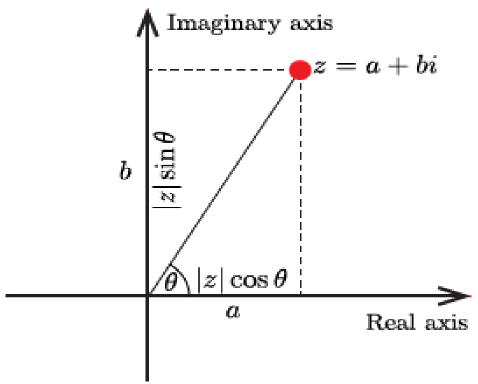
- Complex numbers provide a compact way of describing amplitude and phase (and the operations that affect them, such as filtering)



Complex number $z = x + jy$ (x and y real-valued $j = \sqrt{-1}$.)

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$r = |z| = \sqrt{x^2 + y^2},$$
$$\theta = \arg(z) = \tan^{-1}\frac{y}{x}$$



Kompleks sayının üstel biçimi

- $Z = \cos(Q) + jsin(Q)$
- $\frac{dZ}{dQ} = -\sin(\mathrm{Q}) + jcos(Q)$
- $\frac{dZ}{dQ} = j(\cos(Q) + jsin(Q))$
- $\frac{dZ}{dQ} = j\ Z$
- $\frac{dZ}{Z} = j\ dQ$
- $\int \frac{dZ}{Z} = j \int dQ$
- $Ln\ Z = jQ$
- $Z = e^{jQ}$

Complex Numbers Properties

1. $z+w=w+z$
2. $zw=wz$
3. $\overline{z+w}=\overline{z}+\overline{w}$
4. $\overline{zw}=\overline{z}\overline{w}$
5. $z\overline{z}=\overline{z}z=|z|^2$
6. $\overline{\overline{z}}=z$
7. $|z|=|\overline{z}|$
8. $|zw|=|z||w|$
9. $|z+w|\leq|z|+|w|$
10. $z^{-1}=\dfrac{1}{z}=\dfrac{\overline{z}}{|z|^2}$ when $z\neq 0+0i$

$z=a+bi$ and $w=c+di$.

Imaginary unit number: $i=\sqrt{-1}$

Complex numbers addition: $(a+bi)+(c+di)=(a+c)+i(b+d)$

Complex numbers multiplication: $(a+bi)(c+di)=(ac-bd)+i(ad+bc)$

Complex conjugate: $\overline{z}=a-bi$

Modulus of a complex number: $|z|=\sqrt{z\overline{z}}=\sqrt{a^2+b^2}$

Euler's formula: $e^{i\theta}=\cos\theta+i\sin\theta$

Polar form: $z=|z|e^{i\theta}$

Periodicity of complex numbers: $e^{i\theta\pm 2\pi}=e^{i\theta}$

The Complex Number System

- **Another definitions and Notations**:
- It is the extension of the real number system via closure under exponentiation.

$$i \equiv \sqrt{-1}$$

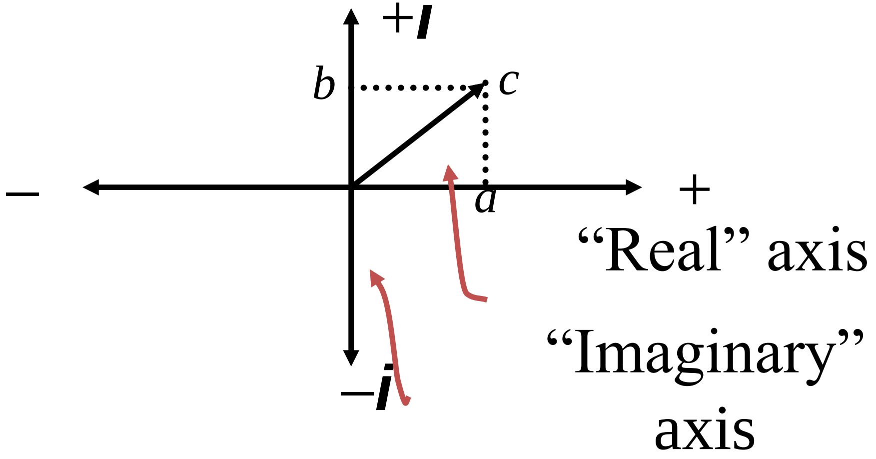
The "imaginary" unit

$$c = a + bi$$

$$\mathcal{Re}[c] \equiv a$$

$$\mathcal{Im}[c] \equiv b$$

$$(c \in \mathbf{C}, a, b \in \mathbf{R})$$



- *(Complex) conjugate:*

$$c^* = (a + bi)^* \equiv (a - bi)$$

- *Magnitude* or *absolute value*:

$$|c|^2 = c^*c = a^2+b^2$$

$$|c| \equiv \sqrt{c^*c} = \sqrt{(a - bi)(a + bi)} = \sqrt{a^2 + b^2}$$

Properties of the polar form

$z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$

1. $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \implies zw = (|z|e^{i\theta})(|w|e^{i\phi}) = |z||w|e^{i(\theta+\phi)}$
2. $(e^{i\theta})^n = e^{in\theta}$, for any number n (i.e., n could be complex!)
3. From the above property $\Rightarrow \dfrac{1}{e^{i\theta}} = (e^{i\theta})^{-1} = e^{-i\theta}$
4. $|e^{i\theta}| = e^{i\theta} \cdot \overline{e^{i\theta}} = e^{i\theta}e^{-i\theta} = e^{i(\theta-\theta)} = e^0 = 1$
5. $\overline{e^{i\theta}} = e^{-i\theta}$
6. Since $e^{\pm 2\pi i} = \cos(\pm 2\pi) + i\sin(\pm 2\pi) = 1$, then $e^{i(\theta \pm 2\pi)} = e^{i\theta} \cdot e^{\pm 2\pi i} = e^{i\theta}$

Complex Numbers

- A complex number **x** is of the form:

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

a: **real part**, b: **imaginary part**

- Addition $(a + jb) + (c + jd) = (a + c) + j(b + d)$

- Multiplication $(a + jb)\,.\,(c + jd) = (ac - bd) + j(ad + bc)$

Complex Numbers

- Multiplication using magnitude-phase representation

$$xy = |x|e^{j\phi(x)} \, . \, |y|e^{j\phi(y)} = |x| \; |y| \; e^{j(\phi(x)+\phi(y))}$$

- Complex conjugate $x^* = a - jb$

- Properties

$$|x| = |x^*|$$
$$\phi(x) = -\phi(x^*)$$
$$xx^* = |x|^2$$

Complex Numbers

- Euler's formula

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

- Properties

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

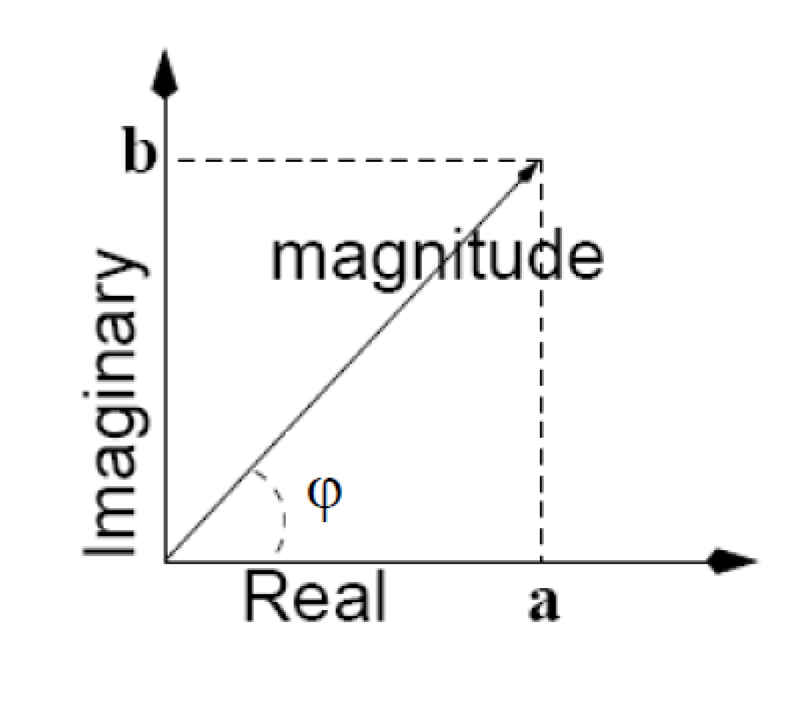
$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Complex Numbers

- Magnitude-Phase (i.e.,vector) representation



Magnitude: $|x| = \sqrt{a^2 + b^2}$

se: $\phi(x) = \tan^{-1}(b/a)$

Phase – Magnitude notation:

$$x = |x|e^{j\phi(x)}$$

Complex numbers

- A complex number $Z = \epsilon C$ is of the form $a, b = \epsilon R$ where $Z = a + ib$ **and *i²=-1***
- Polar representation $Z = Ue^{i\theta}$, where U,θ ϵR

  - *With* $U = \sqrt{a^2 + b^2}$ *the modulus or magnitude*

  - *And the phase* $\theta = \arctan(b/a)$; a ve b'nin işaretlerine bakılarak açının hangi düzlemde olduğu belirlenir (a,b):(+,+),(-,+), (-,-), (+,-).

- *Complex **conjugate***

$$Z = U(\cos\theta + \mathrm{i}\sin\theta) = Ue^{i\theta}$$

$$Z^* = (a + ib\ )^* = Ue^{-i\theta} = U(\cos\theta - \mathrm{i}\sin\theta) = a - ib$$

The Complex Number System

- It is the extension of the real number system via closure under exponentiation.

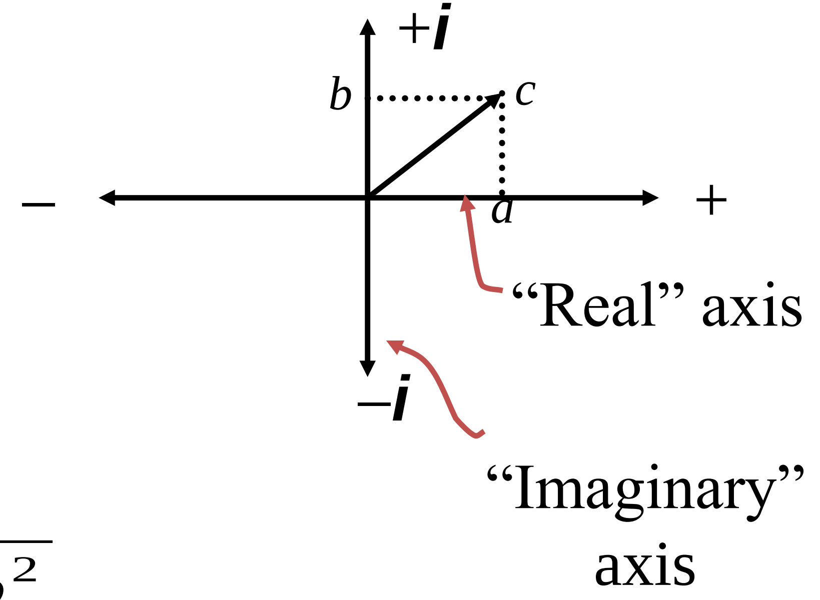
$$i \equiv \sqrt{-1}$$

The "imaginary" unit

$$c = a + bi \qquad (c \in \mathbf{C}, a, b \in \mathbf{R})$$

$$\mathcal{Re}[c] \equiv a$$

$$\mathcal{Im}[c] \equiv b$$



- *(Complex) conjugate:*

  $c^* = (a + bi)^* \equiv (a - bi)$

- *Magnitude* or *absolute value*:

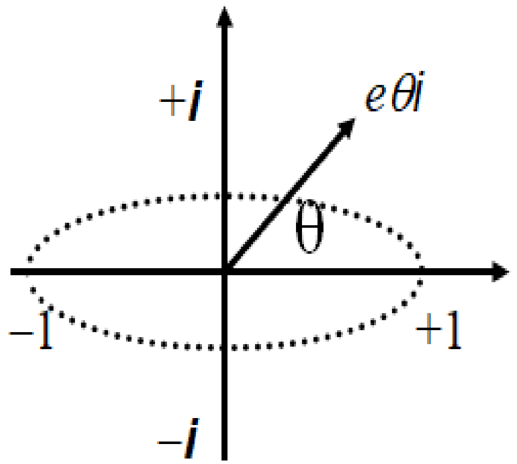
  $|c|^2 = c^*c = a^2+b^2$

$$|c| \equiv \sqrt{c^* c} = \sqrt{(a - bi)(a + bi)} = \sqrt{a^2 + b^2}$$

Complex Exponentiation

- Powers of *i* are complex units:

$$e^{\theta i} \equiv \cos\theta + i\sin\theta$$



- Note:

$$e^{\pi i/2} = i$$

$$e^{\pi i} = -1$$

$$e^{3\pi i/2} = -i$$

$$e^{2\pi i} = e^0 = 1$$

Z1=2 e^πi

Z12 = (2 e πi)^2 = 2^2 (e πi)^2 = 4 (e πi)^2 = 4 e^2πi

Review of complex exponential

geometric series is used repeatedly to simplify expressions.

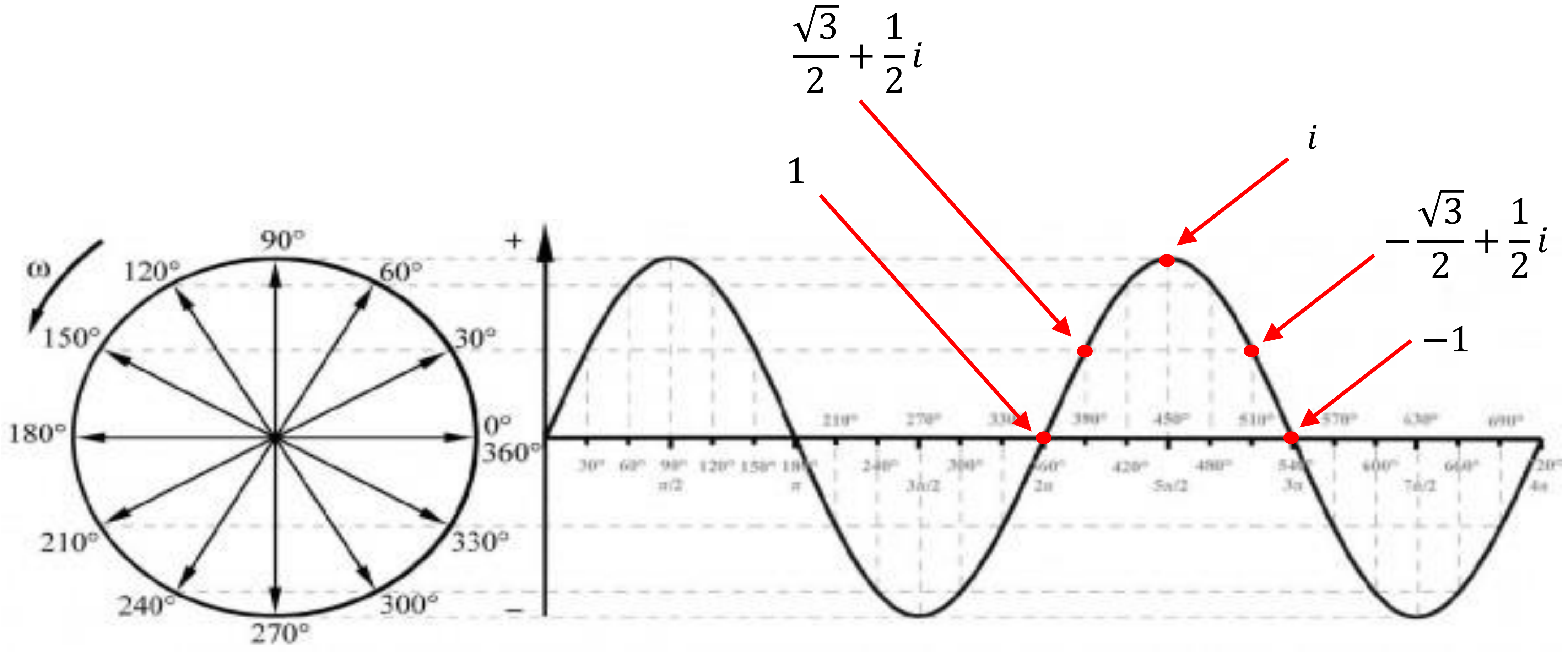
$$\sum_{n=0}^{n-1} x^n = 1 + x + x^2 + \ldots + x^{N-1} = \frac{1-x^n}{1-x}$$

- if the magnitude of x is less than one, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1$$

The geometric series is often a complex exponential variable of the form e^{jk}, where j = $\sqrt{-1}$

Waves


$\frac{\sqrt{3}}{2}+\frac{1}{2}i$
i
1
$-\frac{\sqrt{3}}{2}+\frac{1}{2}i$
-1
ω
90°
120°
60°
150°
30°
180°
0°
360°
210°
330°
240°
300°
270°
+
–

Örnek

What's the polar form of $z = 5 - 5i$? We first need to find the modulus of z, which is given by:

$$
\begin{aligned}
|z| &= \sqrt{5^2 + (-5)^2} \\
&= \sqrt{50}
\end{aligned}
$$

The argument is given by:

$$
\begin{aligned}
\theta &= \arctan\left(\frac{5}{-5}\right) \\
&= \arctan(-1) \\
&= \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}
\end{aligned}
$$

Since the the real part of z is positive and its imaginary is negative,

$$\theta = \frac{7\pi}{4}$$

$$5 - 5i = \sqrt{50}e^{i\frac{7\pi}{4}}$$

Soru

- $Z=A\exp(j\theta)=A*e^{j\theta}=A\cos\theta+J\sin\theta$
- Burada A: Genlik, θ ise radyan cinsinden faz açısını verir.
- **$\phi_{deg}=\phi_{rad}*180/\pi$**
- $J*J=-1$
- $Z=j10\exp(j\pi/2)$ ifadesinde

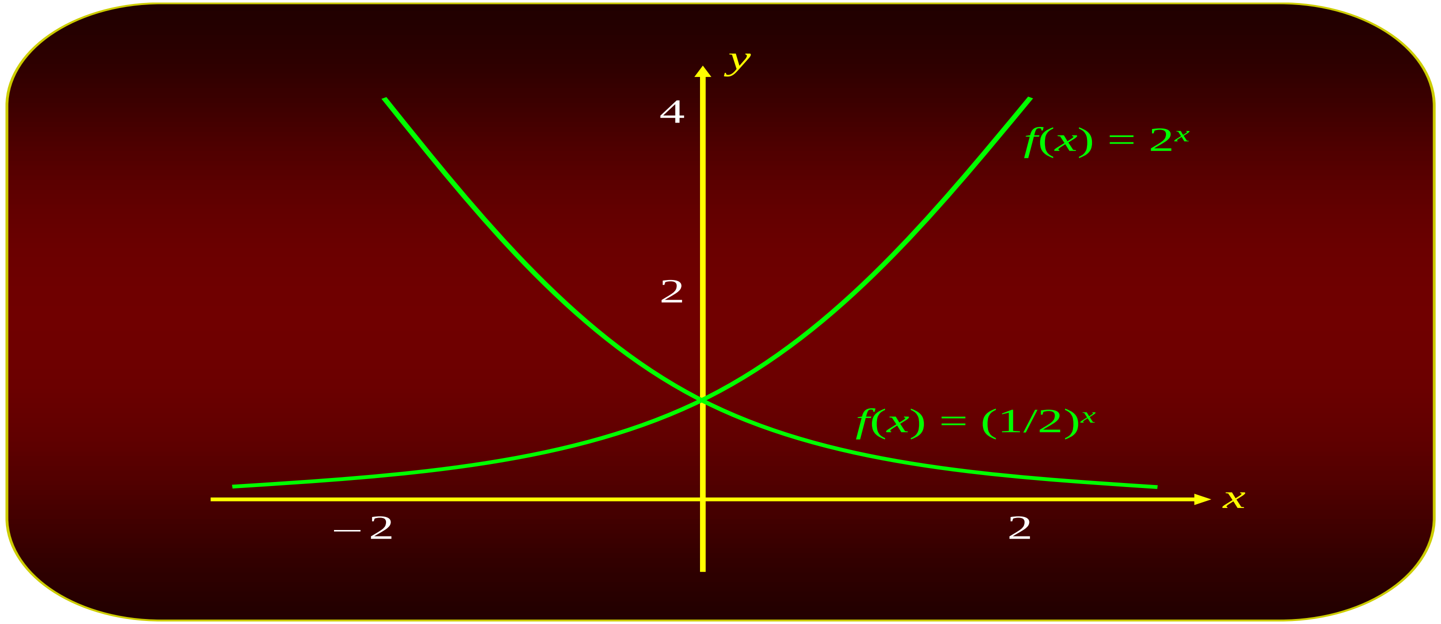
	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Tanımsız	0	Tanımsız	0
cot	Tanımsız	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	Tanımsız	0	Tanımsız

Sorunun Çözümü

- Genlik değeri nedir?
- 10
- Radyan olarak faz açısının değeri nedir?
- $\pi/2$
- Derece olarak faz açısının değeri nedir?
- 90deg.
- Z ifadesini, Z=a+jb cinsinden yazınız.
- Z=j10*(Cos(90)+jsin(90))
- Bir önceki bulduğunuz değeri kullanarak Z ifadesini sadeleştiriniz.
- Z=j10*j=-10

Exponential Function

Exponential Functions



Exponential Function

- The function defined by is called an exponential function with base *b* and exponent *x*.
- The domain of *f* is the set of all real numbers.

$$f(x) = b^x \qquad (b > 0,\ b \neq 1)$$

- The exponential function with base 2 is the function with domain $(-\infty, \infty)$.
- The values of $f(x)$ for selected values of x follow:

$$f(x) = 2^x \qquad f(3) = 2^3 = 8$$

$$f(0) = 2^0 = 1$$

Laws of Exponents

- Let *a* and *b* be positive numbers and let *x* and *y* be real numbers. Then,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $\left(b^x\right)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Examples

- Sketch the graph of the exponential function $f(x) = 2^x$.

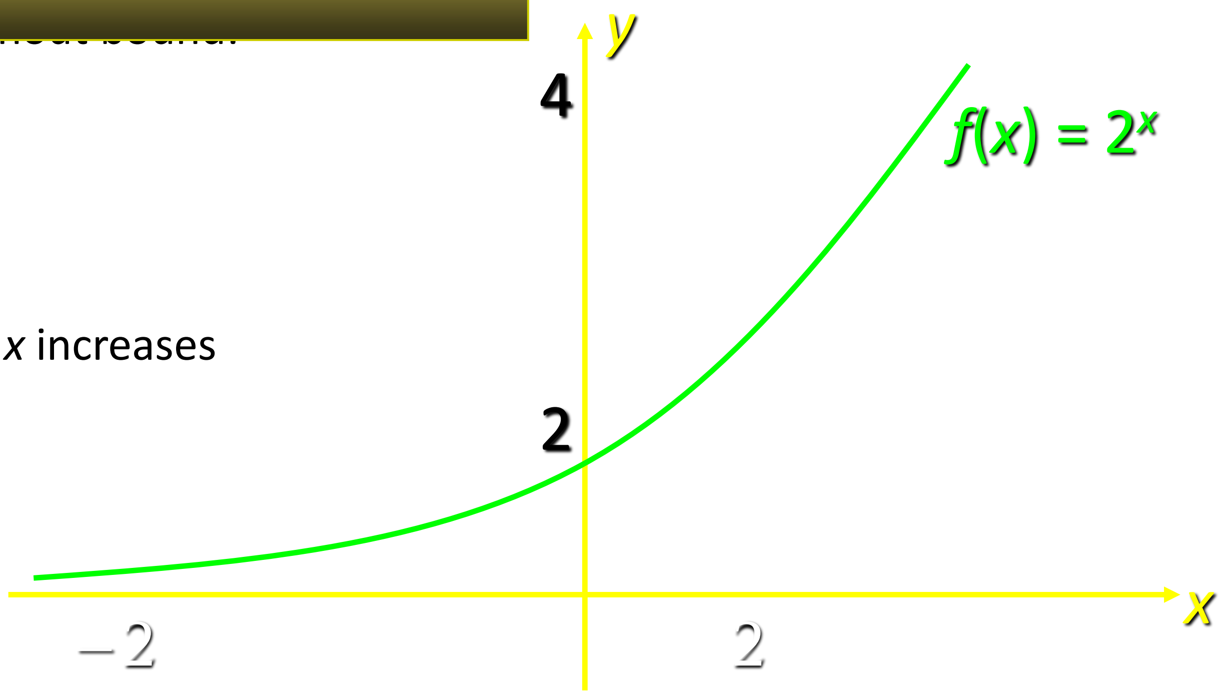
<u>Solution</u>

- Now, consider a few values for x:

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32

  – There is a horizontal asymptote at $y = 0$.

- Furthermore, 2^x increases without bound when x increases without bound.
- Thus, the range of f is the interval $(0, \infty)$.



Properties of Exponential Functions

- The exponential function $y = b^x$ ($b > 0$, $b \neq 1$) has the following properties:

1. Its domain is $(-\infty, \infty)$.
2. Its range is $(0, \infty)$.
3. Its graph passes through the point (0, 1)
4. It is continuous on $(-\infty, \infty)$.
5. It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

The Base *e*

- Exponential functions to the base *e*, where *e* is an irrational number whose value is 2.7182818…, play an important role in both theoretical and applied problems.
- It can be shown that

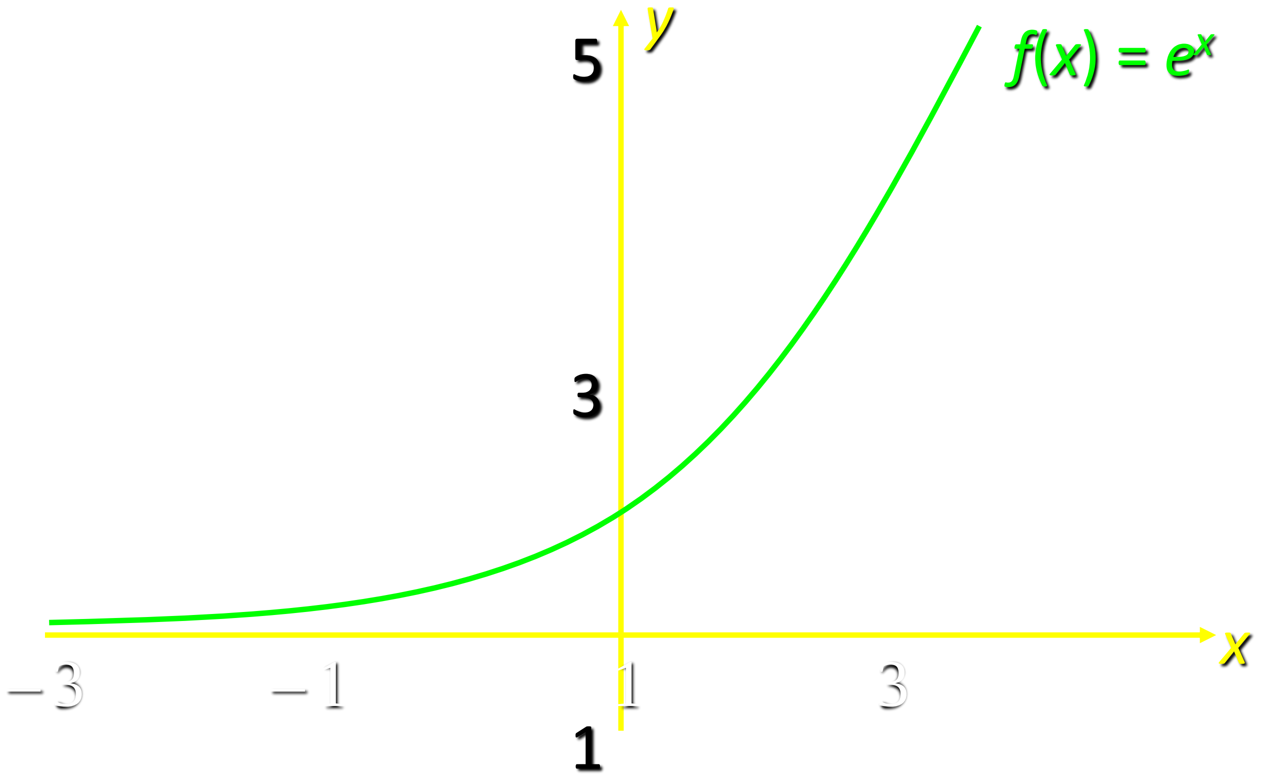
$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

Examples

- Sketch the graph of the exponential function $f(x) = e^x$.

Solution

- Sketching the graph:



Examples

- Sketch the graph of the exponential function $f(x) = e^{-x}$.

Solution

- Since $e^{-x} > 0$ it follows that $0 < 1/e < 1$ and so
  $f(x) = e^{-x} = 1/e^x = (1/e)^x$ is an exponential function with base less than 1.
- Therefore, it has a graph similar to that of $y = (1/2)^x$.
- Consider a few values for x:

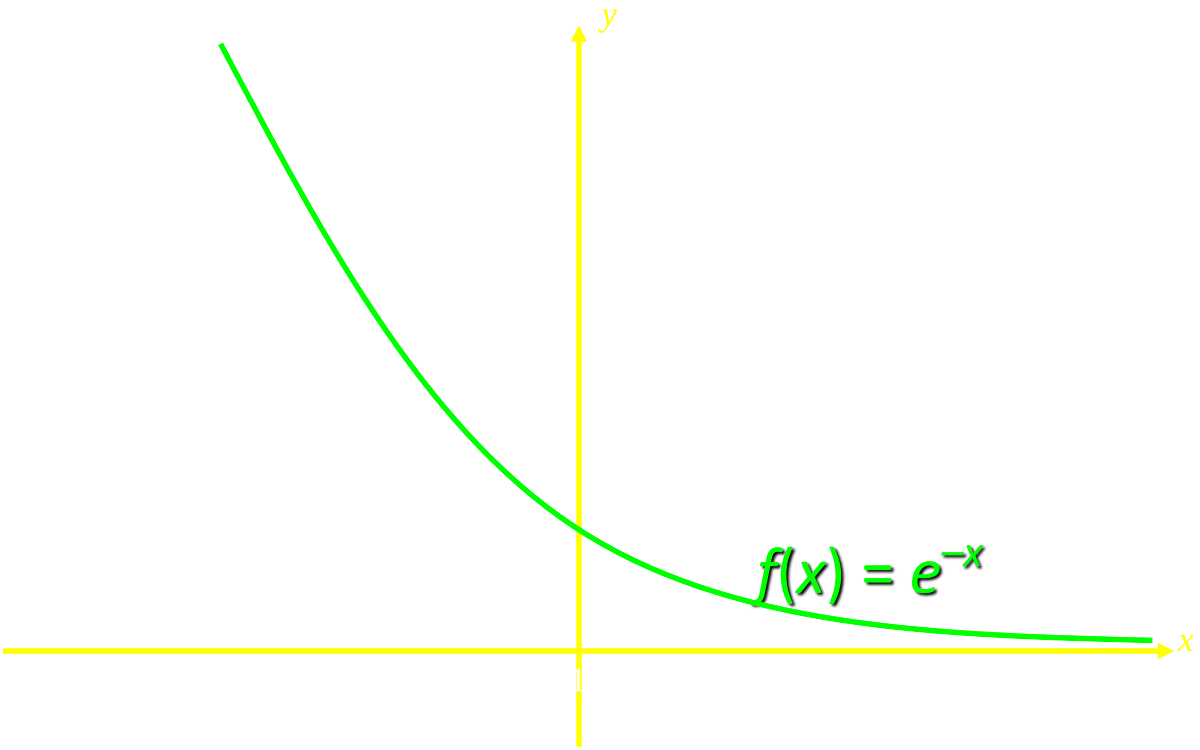
x	– 3	– 2	– 1	0	1	2	3
y	20.09	7.39	2.72	1	0.37	0.14	0.05

Examples

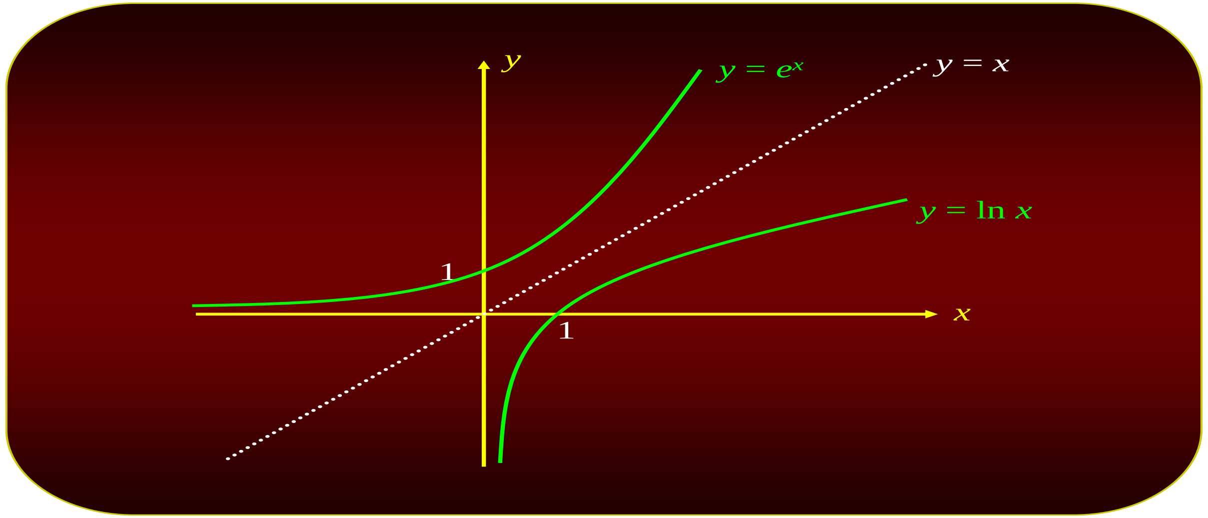
- Sketch the graph of the exponential function $f(x) = e^{-x}$.

Solution

- Sketching the graph:



Logarithmic Functions



Exponential Logarithmic Functions

- Solve the equation $2e^{x+2} = 5$.

<u>Solution</u>

- Divide both sides of the equation by 2 to obtain: $e^{x+2} = \frac{5}{2} = 2.5$

- Take the natural logarithm of each side of the equation and solve:

$$\begin{aligned} \ln e^{x+2} &= \ln 2.5 \\ (x+2)\ln e &= \ln 2.5 \\ x+2 &= \ln 2.5 \\ x &= -2 + \ln 2.5 \\ x &\approx -1.08 \end{aligned}$$

- Properties relating e^x and $\ln x$: $e^{\ln x} = x \quad (x > 0)$

  $\ln e^x = x \quad$ (for any real number x)

Logarithms

- We've discussed exponential equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

- But what about solving the same equation for *y*?
- You may recall that *y* is called the logarithm of *x* to the base *b*, and is denoted $\log_b x$.
  - Logarithm of *x* to the base *b*

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

$$\log_3 x = 4 \text{ implies } x = 3^4 = 81.$$

Examples

- Solve $\log_x 8 = 3$ for x:

<u>Solution</u>

- By definition, we see that $\log_x 8 = 3$ is equivalent to

$$8 = 2^3 = x^3$$
$$x = 2$$

Logarithms

- Exponential equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

- The logarithm of x to the base b, and is denoted $\log_b x$.
- Logarithm of x to the base b

$$y = \log_b x \quad \text{if and only if} \quad x = b^y \qquad (x > 0)$$

$\log x = \log_{10} x$ Common logarithm

$\ln x = \log_e x$ Natural logarithm

$y = \log_b x$ ise $x = b^y$

Laws of Logarithms

- If *m* and *n* are positive numbers, then

$$\log_b mn = \log_b m + \log_b n$$

$$\log_b \frac{m}{n} = \log_b m - \log_b n$$

$$\log_b m^n = n \log_b m$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

Log1=0, Log 2 ≈ 0.3, Log 3 ≈ 0.5, Log 5 ≈ 0.7, Log 7 ≈ 0.8, Log10=1

Examples

- Given that log 2 ≈ 0.3010, log 3 ≈ 0.4771, and log 5 ≈ 0.6990, use the laws of logarithms to find

$$\begin{aligned}\log 15 &= \log 3\cdot 5\\ &= \log 3+\log 5\\ &\approx 0.4771+0.6990\\ &= 1.1761\end{aligned}$$

Logarithmic Function

- The function defined by

$$f(x) = \log_b x \qquad (b > 0,\ b \neq 1)$$

  is called the logarithmic function with base b.
- The domain of f is the set of all positive numbers.

Properties of Logarithmic Functions

- The logarithmic function

  $$y = \log_b x \qquad (b > 0,\ b \neq 1)$$

  has the following properties:

  1. Its domain is $(0, \infty)$.
  2. Its range is $(-\infty, \infty)$.
  3. Its graph passes through the point $(1, 0)$.
  4. It is continuous on $(0, \infty)$.
  5. It is increasing on $(0, \infty)$ if $b > 1$ and decreasing on $(0, \infty)$ if $b < 1$.

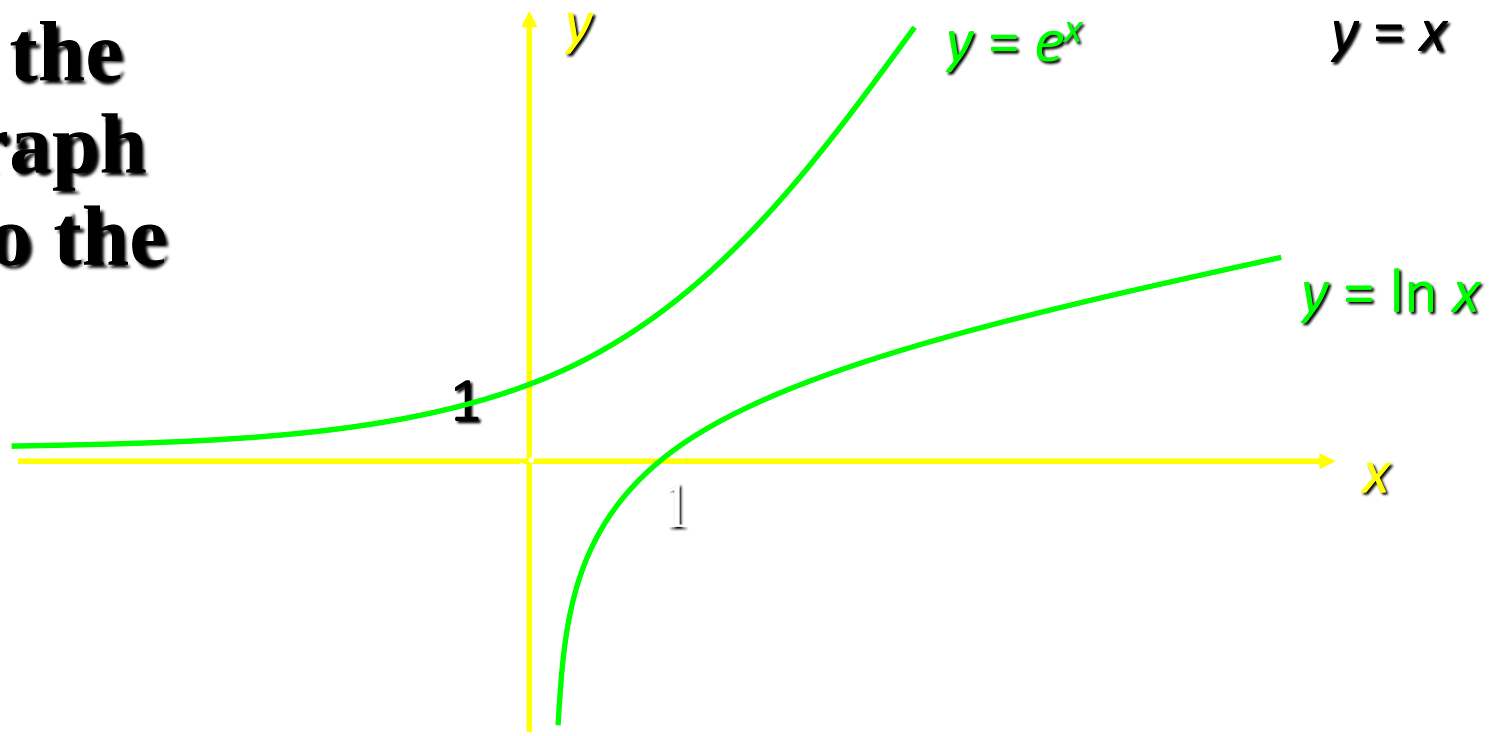
Example

- Sketch the graph of the function $y = \ln x$.

<u>Solution</u>

- We first sketch the graph of $y = e^x$.

- **The required graph is the mirror image of the graph of $y = e^x$ with respect to the line $y = x$:**



Properties Relating Exponential and Logarithmic Functions

- Properties relating e^x and $\ln x$:

$$e^{\ln x} = x \qquad (x > 0)$$

$$\ln e^x = x \qquad \text{(for any real number } x\text{)}$$

Türev

1. $\frac{d}{dx}(c) = 0,$ where c is a constant

2. $\frac{d}{dx}(x^n) = nx^{n-1},$ where n is any real number

Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

3. $\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(e^{cx}) = c\,e^{cx}$

4. $\frac{d}{dx}(\ln x) = \frac{1}{x},$ for $x>0$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \qquad \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

Türevin Yorumu

Birinci Türev

- Birinci ve ikinci türevlerinin verdiği bilgilerden f'(x) veya df/dx olarak yazılan f(x) fonksiyonunun ilk türevi, x noktasındaki teğet çizgisi fonksiyonun eğimidir.
- Grafik olmayan terimlerle ifade etmek gerekirse, ilk türev bize bir fonksiyonun nasıl arttığını veya azaldığını ve ne kadar artacağını veya azalacağını söyler.
- Pozitif eğim bize x arttıkça f(x)'nin de arttığını söyler. Negatif eğim bize x arttıkça f(x)'in azaldığını söyler. Sıfır eğim bize özel bir şey söylemez: fonksiyon o noktada artar, ne azalır veya yerel maksimumda veya yerel minimumda olabilir.

Türevler açısından bu bilgileri yazarken şunu görüyoruz:

-
- $\frac{df(p)}{dx} > 0$, ise f(x), x = p'de artan bir fonksiyondur.
- $\frac{df(p)}{dx} < 0$, ise f(x), x = p'de azalan bir fonksiyondur.
- $\frac{df(p)}{dx} = 0$, ise o zaman x = p, f (x)'in kritik noktası olarak adlandırılır ve x(p)'deki f (x) 'nin davranışı hakkında yorum yapılamaz.

Türevin Yorumu

İkinci Türev

- Bir fonksiyonun ikinci türevi, f’’ (x) veya $\frac{d^2 f}{dx^2}$ olarak yazılır. İlk türev bize fonksiyonun arttığını veya azaldığını söylese de, ikinci türev,
- x = p'de $\frac{d^2 f(p)}{dx^2}$ >0 ise, f(x), x = p'de yukarı doğru kavislidir.
- x = p'de $\frac{d^2 f(p)}{dx^2}$ <0 ise, f(x), x = p'de aşağı doğru kavislidir.
- x = p'de $\frac{d^2 f(p)}{dx^2}$ =0 ise, o zaman f(x) 'in x = p'deki davranışı hakkında bir yorum yapamıyoruz.
-
- Birinci türevin anlamından x, f(x) fonksiyonunun kritik bir noktası olduğunda, o noktada fonksiyonun davranışı hakkında bir yorum yapabilmek için, x’in bölgesel maksimum veya bölgesel minimum olduğunu öğrenmek için genellikle işlevin ikinci türevi kullanılır.

Integral

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} \, dx = \ln x + c$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

$$\int \ln x \, dx = x(\ln x - 1) + c$$

$$\int x e^{ax} \, dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int e^x dx = e^x$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int b^{ax} dx = \frac{1}{a \ln(b)} b^{ax} \qquad ; b > 0$$

$$\int \ln(x) \, dx = x \ln(x) - x$$

$$\int a^x \ln(a) dx = a^x \qquad ; a > 0$$

Dirac δ-'function'

$$\delta(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - \tau)] \, d\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t - \tau) f(t) \, dt = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$

Exponential and Logarithmics Equaations

Properties of Exponential and Logarithmic Equations

Let a be a positive real number such that $a \neq 1$, and let x and y be real numbers. Then the following properties are true:

1. $a^x = a^y$ if and only if $x = y$
2. $\log_a x = \log_a y$ if and only if $x = y$ $(x > 0,\ y > 0)$

Inverse Properties of Exponents and Logarithms

	Base a	*Natural Base e*
1.	$\log_a(a^x) = x$	$\ln(e^x) = x$
2.	$a^{(\log_a x)} = x$	$e^{(\ln x)} = x$

$\ln(2x-3) = \ln 11$	Original Equation
$2x - 3 = 11$	Property of logarithmic equations
$2x = 14$	Add 3 to both sides
$x = 7$	Divide both sides by 2

$4^{x+2} = 64$	Original Equation
$4^{x+2} = 4^3$	Rewrite with like bases
$x + 2 = 3$	Property of exponential equations
$x = 1$	Subtract 2 from both sides

$5 + e^{x+1} = 20$	Original Equation
$e^{x+1} = 15$	Subtract 5 from both sides
$\ln e^{x+1} = \ln 15$	Take the logarithm of both sides
$x + 1 = \ln 15$	Inverse Property
$x = -1 + \ln 15 \approx 1.708$	Subtract 1 from both sides

$5 + e^{x+1} = 20$	Original Equation
$5 + e^{1.708+1} \stackrel{?}{=} 20$	Substitute 1.708 for x
$5 + e^{2.708} \stackrel{?}{=} 20$	Simplify
$5 + 14.999 \approx 20$	Solution checks ✓

Exponential and Logarithmics Equaations

$2^x = 7$ Original Equation

$\log 2^x = \log 7$ Take the logarithm of both sides

$x(\log 2) = \log 7$ Property of Logarithms

$x = \frac{\log 7}{\log 2} \approx 2.807$ Solve for x

$2^x = 7$ Original Equation

$\log_2 2^x = \log_2 7$ Take the logarithm of both sides

$x = \log_2 7$ Inverse Property

$x = \frac{\log 7}{\log 2} \approx 2.807$ Change of Base Formula

$4^{x-3} = 9$ Original Equation

$\log 4^{x-3} = \log 9$ Take the logarithm of both sides

$(x-3)\log 4 = \log 9$ Property of Logarithms

$x - 3 = \frac{\log 9}{\log 4}$ Divide both sides by $\log 4$

$x = 3 + \frac{\log 9}{\log 4} \approx 4.585$ Solve for x

$4^{x-3} = 9$ Original Equation

$\log_4 4^{x-3} = \log_4 9$ Take the logarithm of both sides

$x - 3 = \log_4 9$ Inverse Property

$x - 3 = \frac{\log 9}{\log 4}$ Change of Base Formula

$x = 3 + \frac{\log 9}{\log 4} \approx 4.585$ Solve for x

Series

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \frac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{u^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{u^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

Power series with real variables

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad \text{valid for } -1 < x \le 1$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{valid for all values of } x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all values of } x$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \qquad \text{valid for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{valid for } -1 \le x \le 1$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots \qquad \text{valid for } -1 < x < 1$$

Series

Integer series

$$\sum_1^N n \;= 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = [1 + 2 + 3 + \cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3 + 2.3.4 + \cdots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

Limit

Limit

- Günlük dilde “limit” kelimesi bir miktar, bir fikir ya da herhangi bir şeyin ötesine geçemeyeceği sınırları tanımlamak için kullanılır. Örneğin, hız sınırı size yasal olarak izin verilen maksimum hızı ve kredi kartı limitiniz kullanabileceğiniz maksimum bakiyeyi bildirir. Bu iki örnek de üst sınırları temsil etmektedir. Limit, elbette alt sınırlar için de geçerlidir. Mesela, bir kredi almak için gerekli minimum kredi puanı.

- Matematikte, limit kavramı bir açıdan yukarıdaki örneklerle benzer fakat tam olarak aynı diyemeyiz. Limit, bir sınıra gittikçe yakınlaştığımızda ne olduğu hakkında konuşmak için kullanıldığında aynıdır. Fakat, minimum veya maksimum değerlerle ilgili olmak zorunda olmadığı için de farklıdır. Bunun yerine, matematikte limit fikri ve ele alınan sınırların türü çok daha soyut olabilir.

Limit

- Örnek olarak $1/2^n$ ifadesinde n’ e gittikçe artan tam sayılar verdiğimizde oluşan sayılar dizisini alalım. Diğer bir deyişle n=1, n=2, n=3, n=4 … olduğunda ifadenin alacağı değerleri inceleyelim.
- Kısa bir hesaplama ile dizimizdeki sayıların 1/2, 1/4, 1/8, 1/16 … şeklinde olacağını görürüz.
- Buradaki soru; n’ in çok daha büyük değerlerinde bunu yapmaya devam ederseniz dizi hangi sayıya yaklaşacaktır?
- Diğer bir deyişle n sonsuza giderken dizi asla ulaşamasa bile hangi sayıya yaklaşacak?
- Bu durumda cevabın görülmesi oldukça kolay: n arttıkça dizideki sayıların değeri gittikçe küçülür. n’ in sonsuz büyüklükteki değerlerinin limitinde dizi sıfıra yaklaşır.
- Hiçbir zaman sıfıra ulaşmaz, ama istediğiniz kadar yaklaşır. Gördüğünüz gibi buradaki limit fikri bir önceki geometrik örneğimizden oldukça farklı duruyor. Fakat farklı görünseler bile ikisinin de kalbinde sınırlara yaklaşma konusunda aynı soyut fikir yatıyor.

Belirsizlik

$\frac{\infty}{\infty}$ Belirsizlik Hali

Bu belirsizlik halinde de L' Hospital Kuralı geçerlidir. Zira $\frac{u}{v} = \frac{1}{v} : \frac{1}{u}$ biçiminde yazılabilir. Bu durumda $\frac{\infty}{\infty}$ belirsizliği $\frac{0}{0}$ belirsizliğe dönüşür.

- **$0.\infty$ Belirsizlik Hali**

$u.v = \frac{u}{\frac{1}{v}}$ eşitliği yardımıyla $0.\infty$ belirsizliği $\frac{0}{0}$ veya $\frac{\infty}{\infty}$ haline getirilebilir.

Belirsizlik

- **∞-∞ Belirsizlik Hali**

Bu belirsizlik hali, $u - v = \frac{\frac{1}{v}-\frac{1}{u}}{\frac{1}{uv}}$ eşitliği yardımıyla $\frac{0}{0}$ belirsizlik haline dönüştürebilir.

- **$0^0, \infty^0, 1^\infty$ Belirsizlik Halleri**

x sonlu bir değere veya $\pm\infty$ değerlerine yaklaştığında $y = [u(x)]^{v(x)}$ biçimindeki fonksiyonlar bu belirsizlik hallerinden birini verebilir. Bu durumda her iki tarafın logaritması alınarak

$\ln y = v(x) In\, u(x)$

eşitliği elde edilir. Sağdaki ifadenin limiti, $0.\infty$ belirsizliğine sahip olur. Bu limiti bilinen yolla hesaplanır.

$\lim_{x \to a} \ln y = \lambda \;\; ise \;\; \lim_{x \to a} y = e^{\lambda}$ olur.

0/0 hali

- $\lim\limits_{x \to 3} \frac{2x^2 - 5x - 3}{x^2 + x - 12} = \frac{0}{0}$
- Pay ve paydanın ayrık türevi alınır.
- $\lim\limits_{x \to 3} \frac{4x - 5}{2x + 1} = \frac{7}{7}$ = 1

Example – *Dividing Out Technique*

- Find the limit.

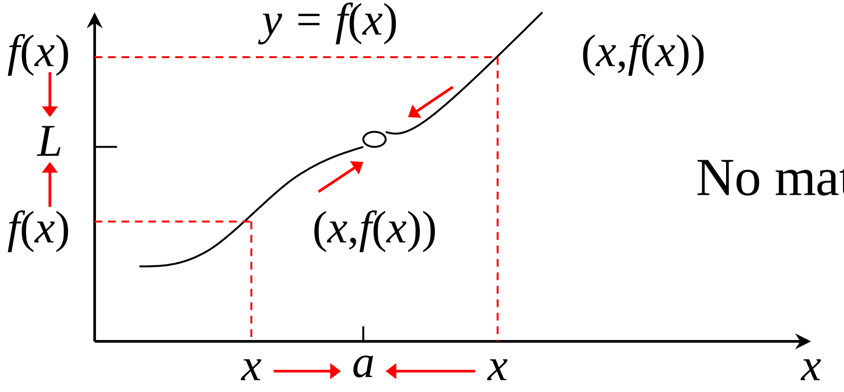
$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

0/0 çıktığı için pay ve paydanın ayrı ayrı türevi alındığında (2x+1)/1=(-6+1)/1= -5

Definition of Limit of a Function

Suppose that the function $f(x)$ is defined for all values of x near a, but not necessarily at a. If as x approaches a (without actually attaining the value a), $f(x)$ approaches the number L, then we say that L is the limit of $f(x)$ as x approaches a, and write

$$\lim_{x \to a} f(x) = L$$



No matter how x approaches a, $f(x)$ approaches L.

Properties of Limits and Direct Substitution

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

Properties of Limits and Direct Substitution

The following summarizes the results of using direct substitution to evaluate limits of polynomial and rational functions.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Theorem

THEOREM

Suppose that $\lim_{x \to a} f(x) = M$ and $\lim_{x \to a} g(x) = N$. Then

$$\lim_{x \to a} \left(f(x) + g(x)\right) = M + N$$

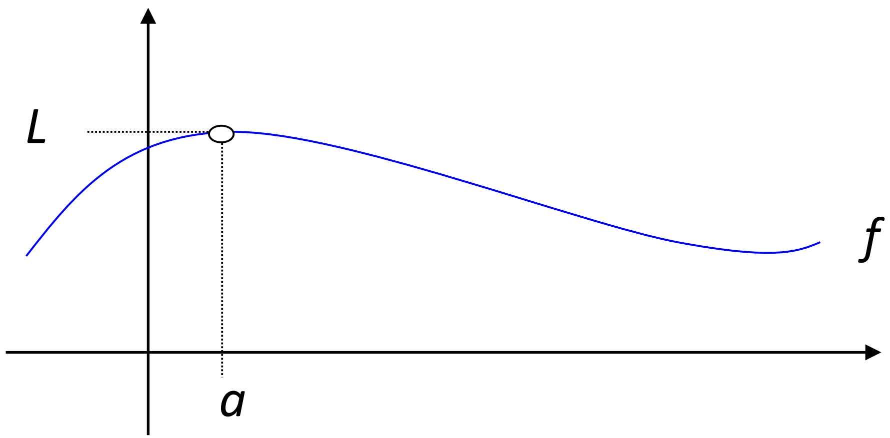
$$\lim_{x \to a} \left(f(x) - g(x)\right) = M - N$$

$$\lim_{x \to a} \left(f(x)g(x)\right) = MN$$

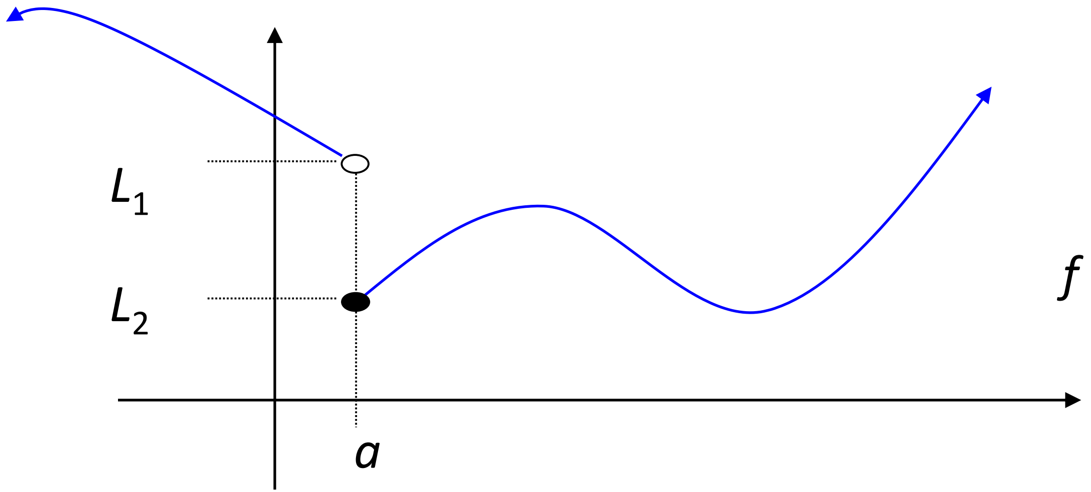
$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{M}{N} \quad (\text{if } N \neq 0)$$

$$\lim_{x \to a} f(x)^k = M^k \quad (\text{if } M, k > 0).$$

Possible Limit Situations



$$\lim_{x \to a} f(x) = L$$

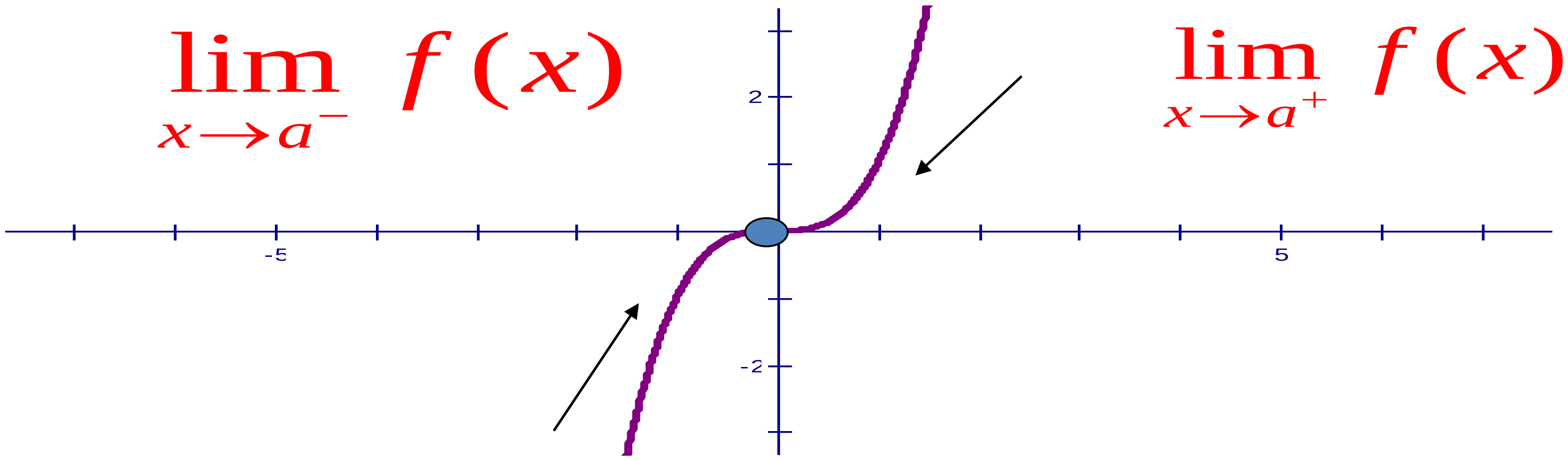


$$\lim_{x \to a} f(x) = DNE$$

DNE = Does Not Exist

Left & Right Hand Limits

$$\lim_{x \to a^-} f(x)$$



$$\lim_{x \to a^+} f(x)$$

Definition: One Sided Limits

Left-Hand Limit: The limit of *f* as *x* approaches *a* from the left equals *L* is denoted

$$\lim_{x \to a^-} f(x) = L$$

Right-Hand Limit: The limit of *f* as *x* approaches *a* from the right equals *L* is denoted

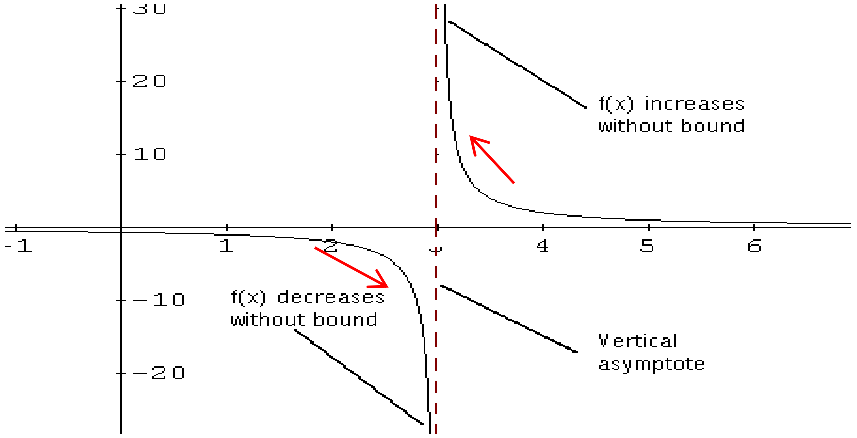
$$\lim_{x \to a^+} f(x) = L$$

Evaluating Limits Graphically

Limits that do not exist

f(x) increases or decreases without bound as *x* approaches *c*.

6. $f(x) = \dfrac{2}{x-3}$



$\lim_{x\to 3} f(x) = dne$

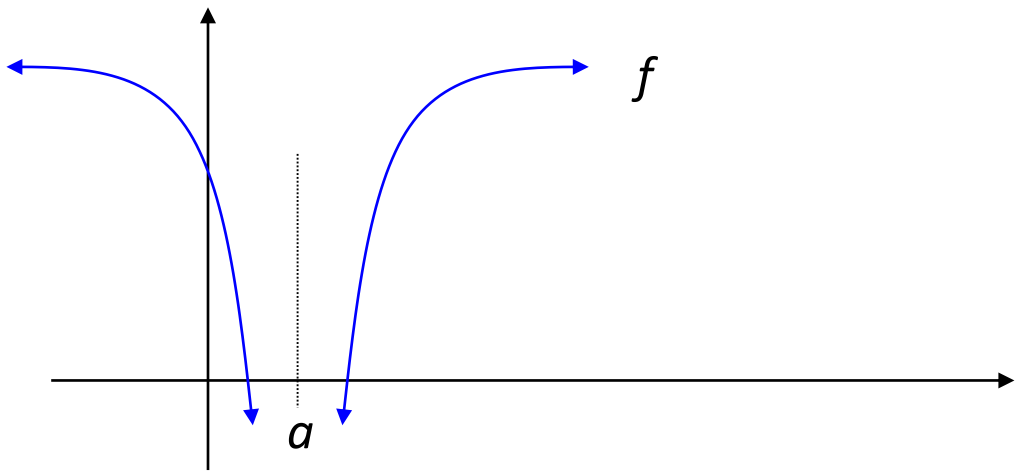
This function is undefined at x = 3, because the denominator goes to zero. It can not be simplified, so there is a vertical asymptote at x = 3.

Approaching 3 from the right, *f(x)* increases without bound.

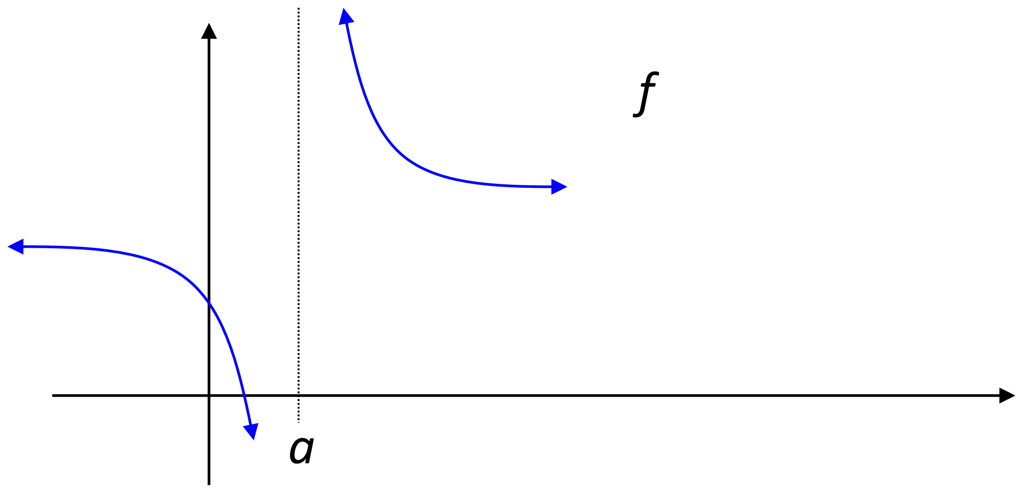
Approaching 3 from the left, *f(x)* decreases without bound.

When the function increases or decreases without bound, the limit does not exist.

Possible Limit Situations



$$\lim_{x \to a} f(x) = -\infty$$



$$\lim_{x \to a} f(x) = DNE$$

DNE = Does Not Exist

THE SUM LAW

The limit of a sum is the sum of the limits.

$$\lim_{x \to a}\left[f(x) + g(x) \right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

THE DIFFERENCE LAW

The limit of a difference is the difference of the limits.

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

THE CONSTANT MULTIPLE LAW

The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$$

THE PRODUCT LAW

The limit of a product is
the product of the limits.

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

THE QUOTIENT LAW

The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

THE POWER LAW

If we use the Product Law repeatedly with $f(x) = g(x)$, we obtain the Power Law.

$$6.\ \lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

where n is a positive integer

USING THE LIMIT LAWS

In applying these six limit laws, we need to use two special limits.

$$7. \lim_{x \to a} c = c$$

$$8. \lim_{x \to a} x = a$$

- These limits are obvious from an intuitive point of view.
- State them in words or draw graphs of $y = c$ and $y = x$.

If we now put $f(x) = x$ in the Power Law and use Law 8, we get another useful special limit.

$$9. \lim_{x \to a} x^n = a^n$$

where n is a positive integer.

Finding a Limit at Infinity

$$\lim_{x\to\infty} \frac{5x^2-7x+1}{2x^2+x+5} = \lim_{x\to\infty} \frac{5-\frac{7}{x}+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{5}{x^2}}$$

$$\lim_{x\to\infty} \frac{5x^2-7x+1}{2x^2+x+5} = \frac{\lim\limits_{x\to\infty} 5-7\cdot\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty} 2+\lim\limits_{x\to\infty}\frac{1}{x}+5\cdot\lim\limits_{x\to\infty}\frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty} 5-\frac{7}{x}+\frac{1}{x^2}}{\lim\limits_{x\to\infty} 2+\frac{1}{x}+\frac{5}{x^2}}$$

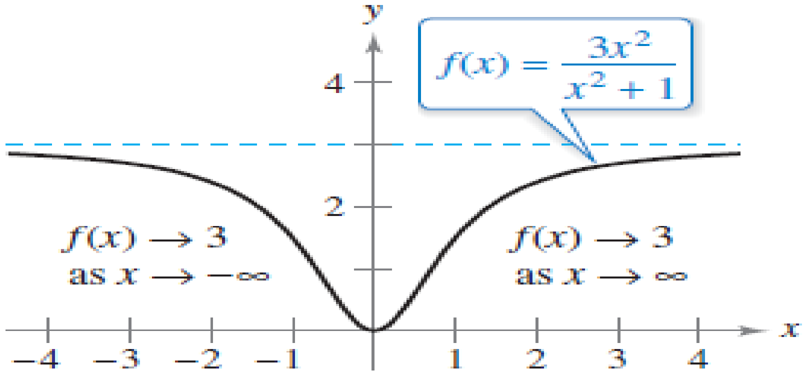
$$= \frac{5-0+0}{2+0+0} = \frac{5}{2}$$

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right) = \lim_{x\to\infty} 5 - \lim_{x\to\infty}\frac{2}{x^2} = 5-0 = 5.$$

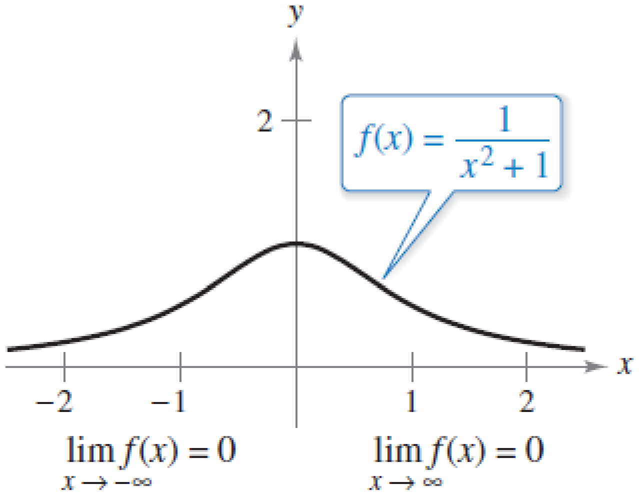
Limits at Infinity

$$f(x) = \frac{3x^2}{x^2 + 1}$$

$$\lim_{x \to -\infty} f(x) = 3 \qquad \lim_{x \to \infty} f(x) = 3.$$



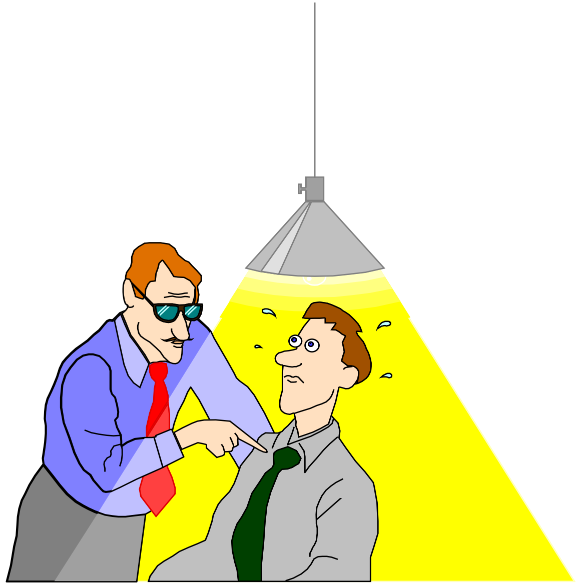
The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.



f has a horizontal asymptote at $y = 0$.

Özümün uğraşısı bir kıvılcım çakmaktır.

Sorular?



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Sincerely,

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thank you